Some Pictures From Cellular Automata Theory

Universidad Adolfo Ibañez

G. Theyssier

LAMA (CNRS, Université de Savoie, France)

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Overview of the talk

1. Cellular Automata
2. Resistance to Noise
3. Mixing and Randomizing
4. Universality

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1. **Cellular Automata**
2. Resistance to Noise
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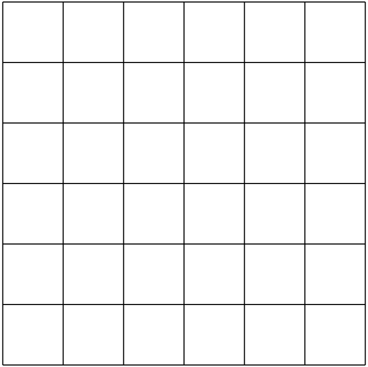
Cellular automata?

Discrete, discrete, discrete...



1. discrete space

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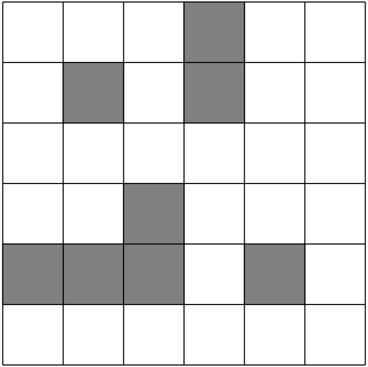

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2. finite set of local states

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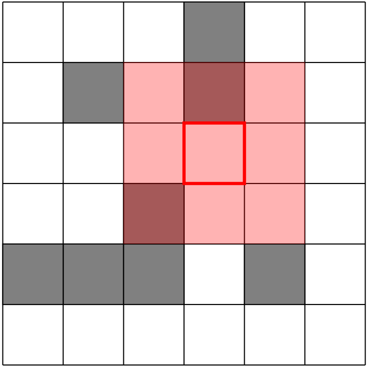
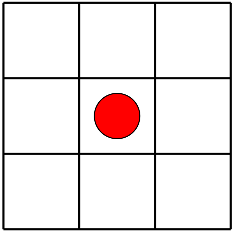


1. discrete space
2. finite set of local states
3. uniform local evolution law at discrete time steps

Example 1: Game of Life



- **states:** dead / alive
- $n =$ nb of alive cells in neighb.
- **birth:** $n = 3$
- **survival:** $n = 3$ or 4
- otherwise **death**

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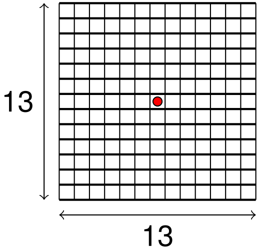
Example 2: Majority



- **states:** 0 and 1
- **rule:** change to majoritary state as seen in neighborhood

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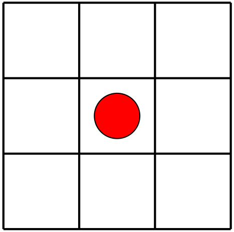
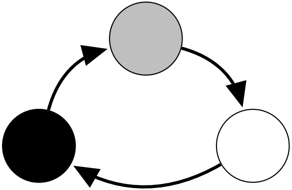
Can you guess the global behavior?





rule: change to next state in the cycle **if** seen $\geq$ 3 times in neighborhood, **otherwise** do not change

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1. does $\lim \mu_t(x)$ depend on x?
2. can we do something useful with a ϵ-pertubated rule?

Experiments

ϵ-perturbations with $\epsilon = 0.05$

- Game of Life

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- Majority

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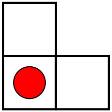
- Toom's majority



rule: change to majoritary state as seen in neighborhood

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Theorems

Toom's majority

Toom's rule is not ergodic.

Reliable computation

For any 1D CA F, there is a 3D CA G that can simulate F even with ϵ-perturbation

(for ϵ small enough)

P. Gács *et. al.*

http://www.cs.bu.edu/~gacs/recent-publ.html

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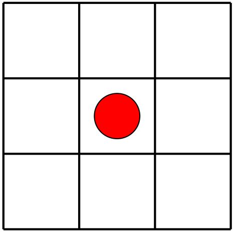
1. what is $\lim \mu_t$?
2. what kind of convergence?

Experiments

The sum modulo rule



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- **states:** $\{0, \ldots, p-1\}$
- **rule:** $\phi(x) = \sum x_i \bmod p$

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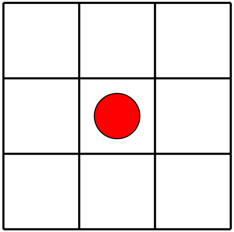

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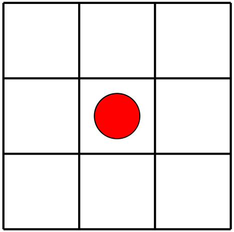

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- **(p=7)** looking at μ_t when starting with many 0s

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Theorem

- **Cesaro mean:**

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Averaging

Starting from any distribution of states*, M_t converges to uniform distribution

- a kind of second law of thermodynamics

M. Pivato *et. al*

http://euclid.trentu.ca/pivato/Research/research.html

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Different Notions of Universality

- Cellular Automata can be simulated on a computer

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- **Difficulty:** how to define “can simulate”?

Simulation of a CA by another CA

- **intuition:**
  - **simulated** / simulator
  - **1 cell** $\leftrightarrow$ $m \times n$ block of cells
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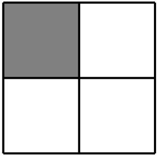
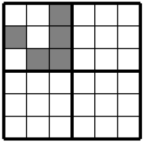
- **example:**

Diagonal shift



T=0

Game of Life



T=0

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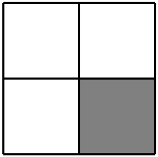
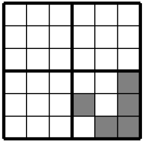


T=1

Game of Life



T=12

Intrinsic Universality

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Theorem

For symmetric CA, the proportion of universal CA goes to 1 when size (states or neighborhood) goes to ∞

- symmetric vs. non-symmetric

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¡Gracias!