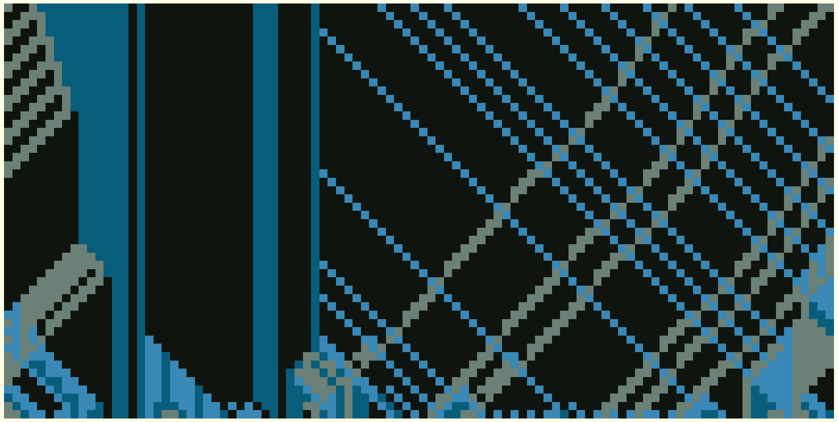
Captive Cellular Automata



MFCS 2004, Praha

Guillaume Theyssier (LIP, ENS Lyon, France)

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local definition $\stackrel{?}{\longrightarrow}$ global dynamics

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Example of additive CA (Martin et al., 1984)

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⇒ a new attempt : Captive Cellular Automata (CCA)

A cellular automaton $\mathcal{A}$ is a 4-uple $(\mathbb{Z}^d, N, S, \delta)$:

$\mathbb{Z}^d$	lattice of cells
$N = \{\overrightarrow{n_1}, \ldots, \overrightarrow{n_k}\}$	vectors of $\mathbb{Z}^d$, neighbourhood of $\mathcal{A}$
S	a finite set of states
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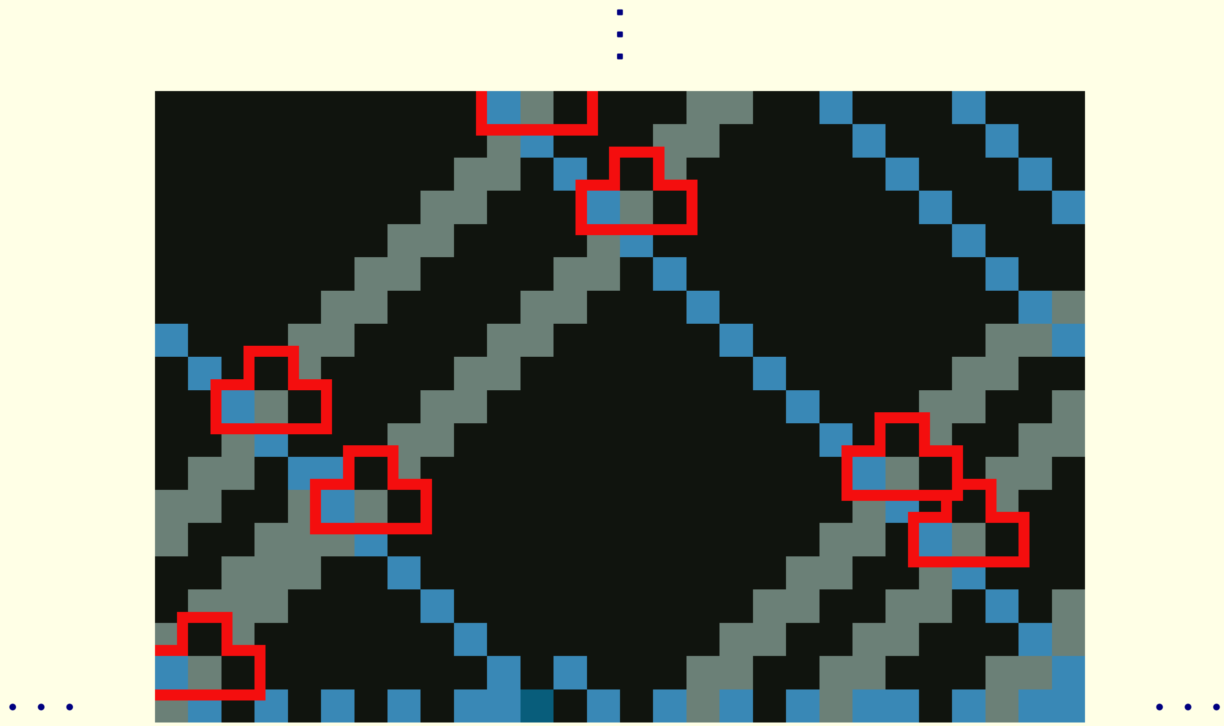
- A global mapping on configurations is obtained by uniform and synchronous application of δ:

$$\forall c \in S^{\mathbb{Z}^d}, \forall \overrightarrow{z} \in \mathbb{Z}^d : \mathcal{A}(c)_{\overrightarrow{z}} = \delta(c_{\overrightarrow{z}+\overrightarrow{n_1}}, \ldots, c_{\overrightarrow{z}+\overrightarrow{n_k}})$$

Space-time diagram

$$\mathcal{A} = (\mathbb{Z}, N = \{-1, O, 1\}, S = \{\blacksquare, \blacksquare, \blacksquare, \blacksquare\}, \delta)$$



(time goes from bottom to top)

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"Up to renaming, $\mathcal{B}$ is $\mathcal{A}$ restricted to a subset of states."

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Remarks :

- a property of the local transition map
- a CA with 2 states is captive if and only if its 2 states are quiescents
- a CCA with a neighbourhood of size n is entirely determined by its n-states sub-automata

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classical algorithmic constructions to be revisited

(e.g. simulating larger radius with more states)

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Remark: set conserving $\Rightarrow$ captive (the converse is false).

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Proposition : there is exactly 1 expansive CCA with radius 1

$$\mathcal{A}(c)_i = c_{i-1} + c_i + c_{i+1} \bmod 2 \text{ on states set } \{0, 1\}$$

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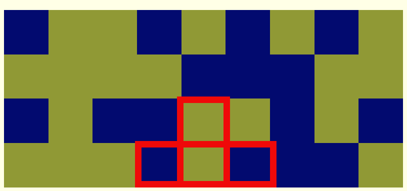
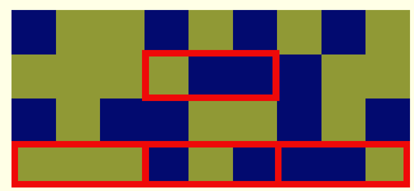


states: A^m



$$\mathcal{A} \xrightarrow{\vec{T}=<m,n>} \mathcal{A}^{\vec{T}}$$

$$\mathcal{A}^{<m,n>} = o^m \circ \mathcal{A}^n \circ o^{-m}$$

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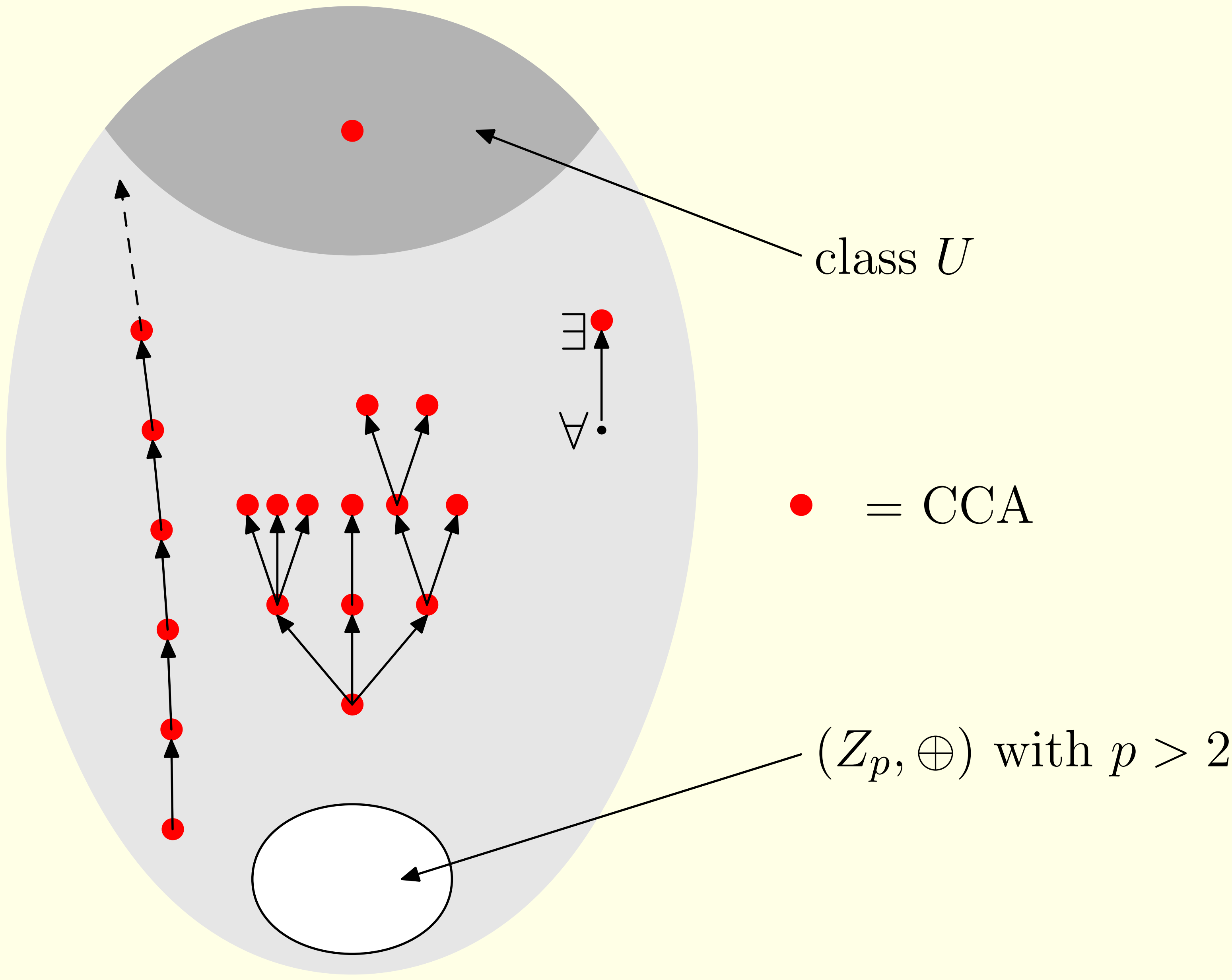
Where are CCA in the ordered structure $($CA$/\sim, \preceq)$?


class U
$\exists$
$\forall$
= CCA
$(Z_p, \oplus)$ with $p > 2$

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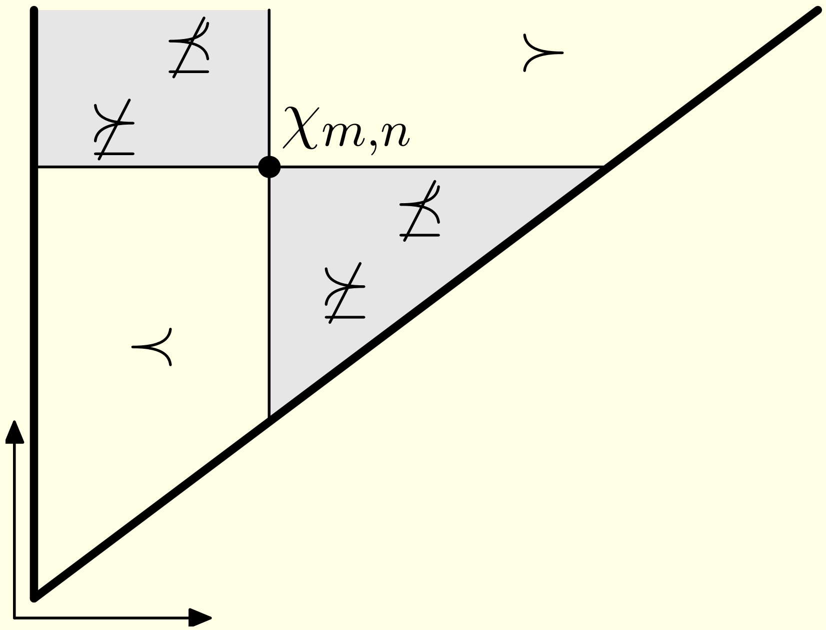
Proposition: $\exists$ a familly $(\chi_{m,n})_{n\geq m}$ of CCA s.t.



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"An odd number of states appear in the limit set" is a non-trivial property (for CCA) which is decidable (for CCA).

No more "Rice theorem" for properties of limit sets, but...

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The proof rely on the ability for CCA to uniformly simulate CA.

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input a CA $\mathcal{B}$ with $r_{\mathcal{B}} = r_{\mathcal{A}}$

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P1 is undecidable whereas P2 is decidable.

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- complexity hierarchy according to neighbourhood for CCA?
- what are $\sim$-classes of CA avoided by CCA?