Pre-Expansivity in Cellular Automata

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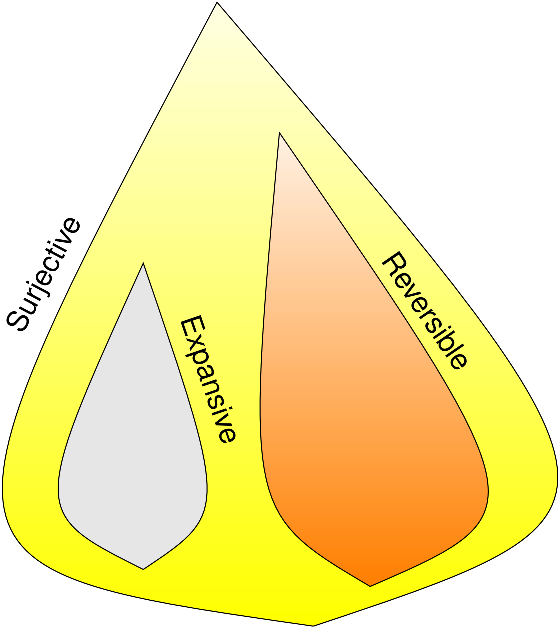


■ DPO conjecture /decidability

Surjectivity

Garden of Eden Theorem

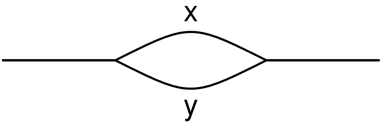
F surjective **iff** F **pre-injective**

- Theorem true for a large class of lattices
- **Asymptotic configurations:** $x \overset{*}{=} y$

  equal except on a finite number of cells

- **pre-injectivity** $\equiv$ injectivity for asymptotic configurations

$$x \overset{*}{=} y \Rightarrow (x \neq y \Rightarrow F(x) \neq F(y))$$



- **finite differences never disappear**

Expansivity

- **Definition:**

$$\exists \epsilon, \forall \mathbf{x}, \forall \mathbf{y},\ \mathbf{x} \neq \mathbf{y} \Rightarrow \exists t, d\big(F^t(\mathbf{x}), F^t(\mathbf{y})\big) > \epsilon$$

- ***k*-trace** $\mathcal{T}_k : Q^{\mathbb{Z}^d} \to \left(Q^{[-k,\dots,k]^d}\right)^{\mathbb{N}}$

$$\mathcal{T}_k(c) = t \mapsto F^t(c)_{[-k,\dots,k]^d}$$

- **expansivity** $\Leftrightarrow$ there is k s.t. $\mathcal{T}_k$ is injective

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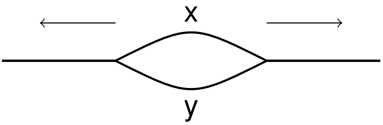
Facts

- there is no reversible expansive CA
- there is no 2D expansive CA

Pre-expansivity

- **Definition:** expansivity but only for asymptotic pairs

$$\exists\epsilon, \forall\mathbf{x}, \forall\mathbf{y},\ \mathbf{x} \neq \mathbf{y} \wedge x \overset{*}{=} y \Rightarrow \exists t, d\big(F^t(\mathbf{x}), F^t(\mathbf{y})\big) > \epsilon$$



"finite differences spread over the lattice"

- pre-**expansivity** $\Leftrightarrow$ there is k s.t. $\mathcal{T}_k$ is pre-injective
- expansivity $\Rightarrow$ pre-expansivity $\Rightarrow$ surjectivity

For linear CA

pre-expansivity $\Leftrightarrow$ expansivity on finite configurations

Overview of the talk

1. Dimension 1

2. Dimension 2

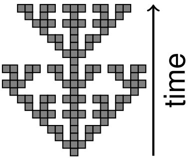
Overview of the talk

1. **Dimension 1**

2. Dimension 2

Expansivity

- simple example: $f(x_{-1}, x_0, x_1) = x_{-1} + x_0 + x_1 \bmod 2$

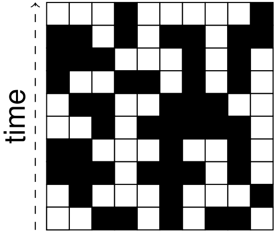


- any LR-permutive CA is expansive
- decidable for linear CA
- general decidability still open...

What about pre-expansivity?

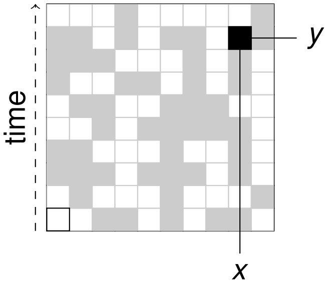
Dependencies

- fix initial configuration c



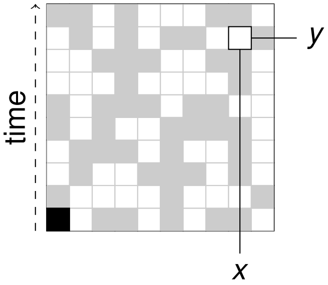
Dependencies

- fix initial configuration c
- observe cell x after y steps



Dependencies

- fix initial configuration *c*
- observe cell *x* after *y* steps
- change initial state of cell 0



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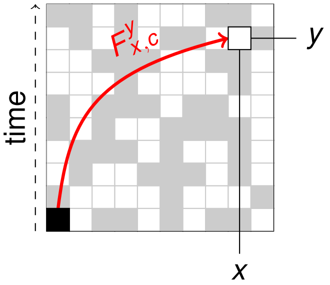

Dependency function

$$F^y_{x,c} : Q \to Q$$

Isolated dependencies

Two extremal cases

- $F^{y}_{x,c}$ bijective (full-dependency)
- $F^{y}_{x,c}$ **constant** (no dependency)

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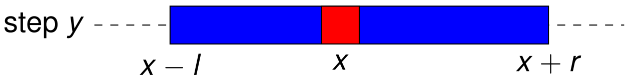
Property Spot[x,y,l,r]

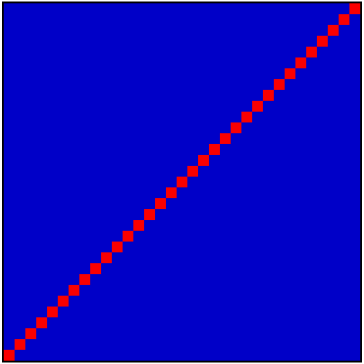
For all configurations c:

- $F^y_{x,c}$ is a bijection;
- $F^y_{z,c}$ is **constant** for all $z \in [x - l; x + r] \setminus \{x\}$

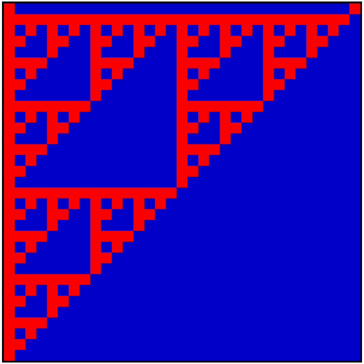


- **Remark:** when F linear, it is enough to consider $c = \overline{0}$

Examples



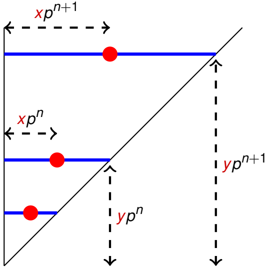
Shift



$x + y$ mod 2

Self-similar dependency structures

- fix F and $p \geq 2$
- geometric progression of isolated dependencies



Definition

$(x, y) \in X_p \subseteq \mathbb{R} \times \mathbb{R}_+$ *iff*

$$Spot[xp^n, yp^n, p^{n-k}, p^{n-k}]$$

for some k and for all large enough n

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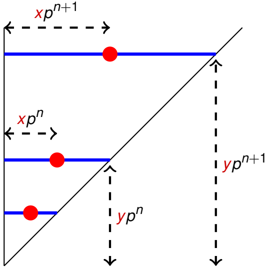

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for some k and for all large enough n

- **Proposition:** $X_p(F) \neq \{(0,0)\} \;\Rightarrow\; F$ surjective

X_p vs. pre-expansivity

Proposition

If there is $(x_1, y_1) \in X_p$ with $x_1 > 0$ and $(x_2, y_2) \in X_p$ with $x_2 < 0$ **then** F is pre-expansive.

- **intuition:** larger and larger isolated dependencies allows to “transport” larger and larger finite words

- how to prove such a property of X_p?

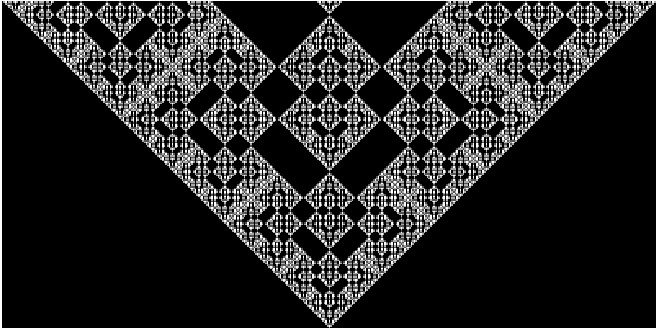
J. Gütschow, V. Nesme, R. F. Werner

Self-similarity of Cellular Automata on Abelian Groups

A concrete example

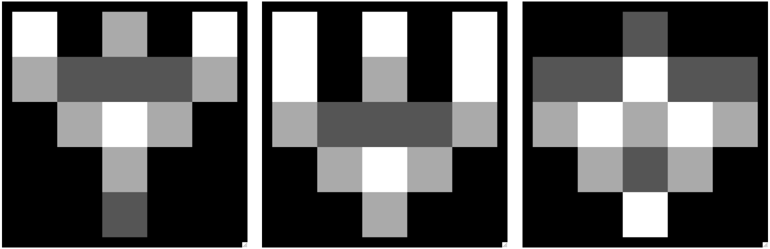
- $Q = \{0,1\} \times \{0,1\}$

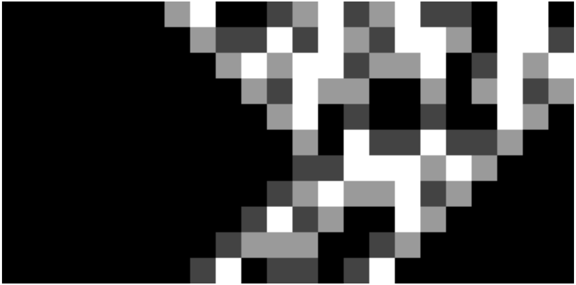
$$\Theta = \begin{pmatrix} 0 & 1 \\ 1 & X^{-1} + 1 + X \end{pmatrix}$$



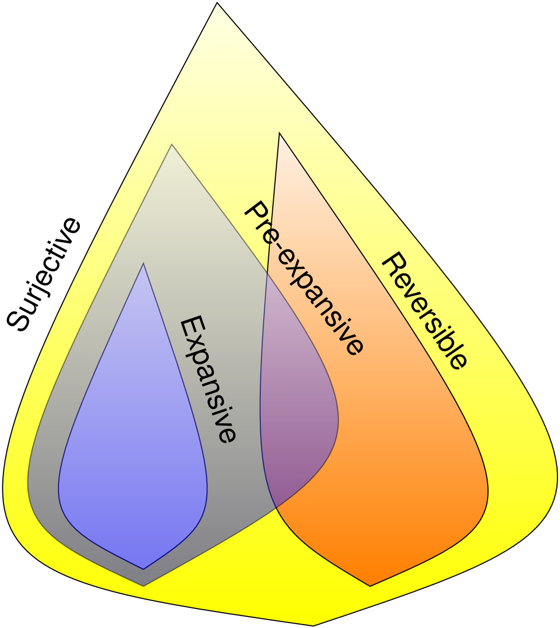
Reversible, not expansive but **pre-expansive**

A concrete example





Situation in 1D



Overview of the talk

1. Dimension 1

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Spoiler

Does there exist a
2D pre-expansive CA?

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Does there exist a
2D pre-expansive CA?

I don't know!

- no obvious entropy obstruction
- no obvious rule achieves it
- let's look closer...

Restriction to simple linear CA

- $Q = \{0, \ldots, p-1\}$, p prime
- $F(c)_0 = \sum_{v \in V} c_v \bmod p$

- $\mathcal{T}_k$ is then a linear application
- $\left(T_z^k\right)_{z \in \mathbb{Z}^2}$ set of traces of 'spot' configurations

$$T_z^k = \mathcal{T}_k(c^z)$$

where $c^z(z') = 1$ if $z = z'$, 0 else.

Fact

F pre-expansive **iff** $\exists k$ no finite linear combination of T_z^k is $\overline{0}$

Amplification Lemma

Amplification

If there is c finite with $\mathcal{T}_k(c) = \overline{0}$ and $k \geq |v|$ $(\forall v \in V)$
then for any k' there is c' with $\mathcal{T}_{k'}(c') = \overline{0}$ and $|c|_{\neq 0} = |c'|_{\neq 0}$

- proof via Lucas' Theorem

$$\binom{p^i}{n} \neq 0 \text{ mod } p \Rightarrow n = 0 \text{ or } n = p^i$$

- c' of the form

$$c'(x,y) = \begin{cases} c(a,b) & \text{if } (x,y) = (p^i a, p^i b) \\ 0 & \text{else.} \end{cases}$$

Amplification Lemma

Consequences

- enough to study $\left(T_z^k\right)$ for some fixed k

- $x \overset{m}{=} y \Leftrightarrow x$ and y differ on **exactly** m positions
- m-prexpansivity: $x \overset{m}{=} y \Rightarrow \mathcal{T}_k(x) \neq \mathcal{T}_k(y)$

- $E = \{m : F \text{ is not } m\text{-prespansive}\}$
- E is stable under addition

Alternative

- either only finitely many m such that F m**-prexpansive**
- or F m**-prexpansive** for all m except (possibly) multiples of some fixed prime number

Regularity of traces

Hypothesis H

There is a linear function λ s.t. T_z^k is determined by its prefix of size $\lambda(\|z\|_\infty)$

- **True** for $p = 2$ and V von Neumann or Moore
  - if $z < 2^k$, take $u = T_z^k[1 \cdots 2^k]$ and $v = T_z^k[2^k + 1 \cdots 2^{k+1}]$
  - T_z^k is the fixed-point of substitution

$$u \rightarrow uv$$
$$v \rightarrow uu$$

- Believed true for a large class of linear CA (via substitution)

Positions vs. traces

Proposition

If hypothesis H + amplification lemma hold
then F is **not** pre-expansive

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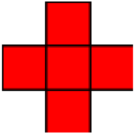
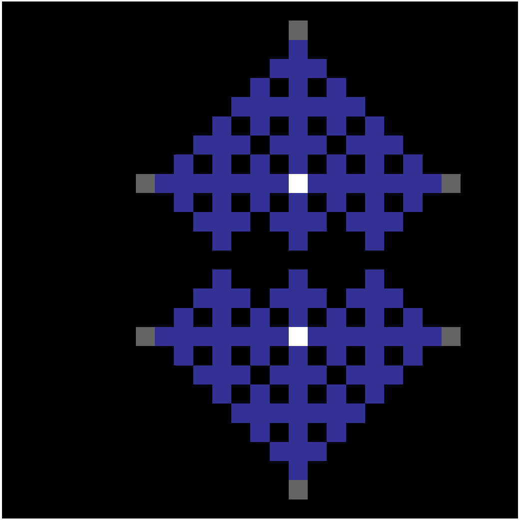
Proposition

If hypothesis H + amplification lemma hold
then F is **not** pre-expansive

- k fixed
- **quadratic** number of points z in a ball of radius n
- prefix of **linear** size determines trace
- $Vec\{T_z^k\}_{\|z\|_\infty \leq n}$ is of dimension linear in n
- so for n large enough, $\left(T_z^k\right)_{\|z\|_\infty \leq n}$ are linearly dependent

Concrete examples





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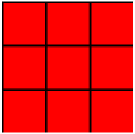

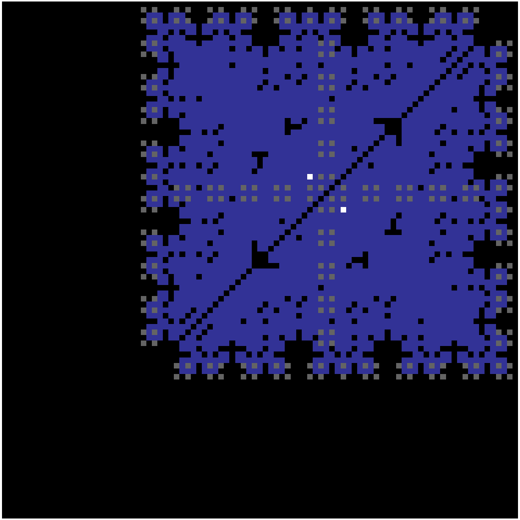

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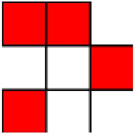

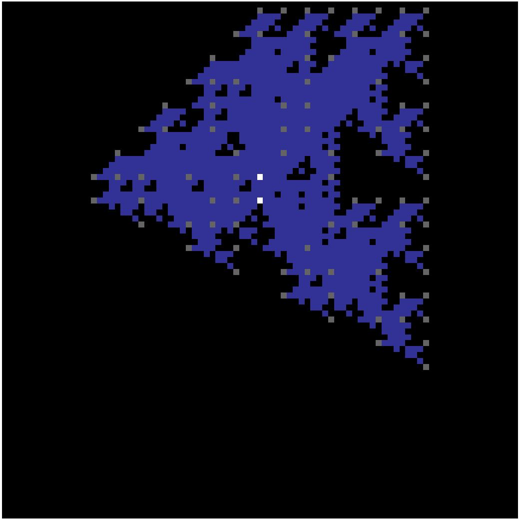

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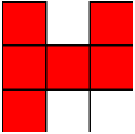

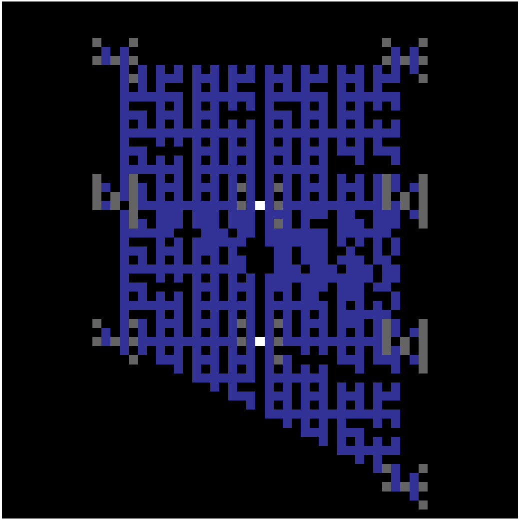

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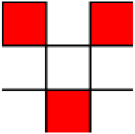

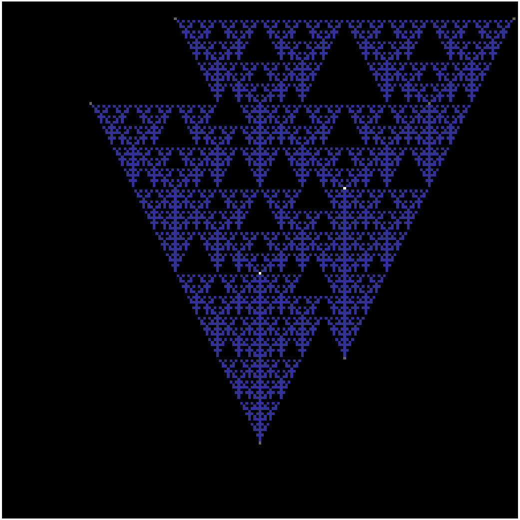

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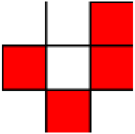

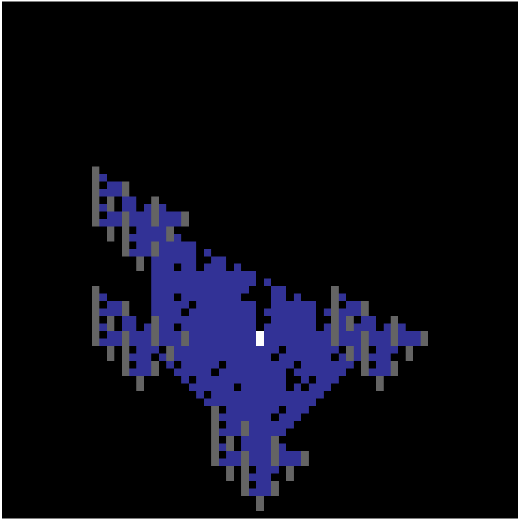

A positive result

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F with $p = 2$ and V von Neumann is 3-prexpansive

- substitution $u \to uv$ / $v \to uu$
- first non-zero time in traces

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- first non-zero time in traces

- **expected**: F is $(2n+1)$-prexpansive for all n

Questions?