Subshifts and MSO Logic

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Overview

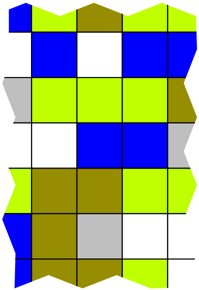
- **Two worlds**:
  1. *symbolic spaces*: words, tilings, subshifts, etc
  2. *MSO logic*

- **Two kinds of results**:
  1. logic characterisation of some families of subshifts
  2. 'combinatorial' characterisation of some classes of formulas

- **Focus of this talk**:

  how classical results on words and pictures extend to sofic subshifts

General Setting



Symbolic space

- a regular domain D
- a finite alphabet Q
- objects are configurations
  i.e. mappings $D \to Q$

MSO logic

- *FO variables*: positions in D
- *SO variables*: subsets of D
- *unary functions*: elementary displacements in D
- *unary predicates*: colouring

$$\exists X, \forall z, \quad P_{\blacksquare}(z) \implies X(\mathrm{East}(z))$$

Model Theoretical Approach

Formulas and models

- an object $\mathcal{M} : D \to Q$
- an MSO formula ϕ
- $\mathcal{M}$ models ϕ if *[...usual def...]*

Definability and equivalence

- ϕ defines the set of its models
- ψ and ϕ are equivalent if they define the same sets

MSO fragments

- EMSO $\stackrel{def}{=}$ formulas of the form $\exists X \phi(X)$ where ϕ has only FO quantifiers
- SO quantifier alternation hierarchy: $\Sigma_1^{SO} = \text{EMSO}$
- Within EMSO, FO quantifier alternation hierarchy

First Order: Locality and Thresholds

Threshold counting of finite patterns

- P: finite pattern
- k: threshold
- $S_{=k}(P) \overset{def}{=}$ configurations with **exactly** k occurrences of P
- $S_{\geq k}(P) \overset{def}{=}$ configurations with **at least** k occurrences of P

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Theorem

Every FO definable set is a positive combination (unions and intersections) of sets of type $S_{=k}(P)$ or $S_{\geq k}(P)$.

Idea: Hanf locality lemma adapted to this setting

Classical results

Dimension 1: words

Th. (Büchi 60, Elgot 61): a language is regular iff it is MSO definable.

Th. (Thomas 82): over words, every MSO sentence is equivalent to a 1-EMSO sentence.

Key idea: finite automata and their closure properties

Classical results

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Key idea: finite automata and their closure properties

Dimension 2: finite pictures

Th. (Giammarresi et al. 94): a picture language is recognizable iff it is EMSO definable.

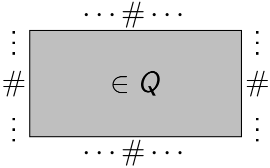
Th. (Matz, Thomas 97): the SO alternation hierarchy over pictures is infinite.

Recognizable? Picture?

Focus on Pictures

Pictures:

- new alphabet: $Q \cup \{\#\}$
- picture $\stackrel{def}{=}$ rectangular Q-pattern surrounded by $\#$ states



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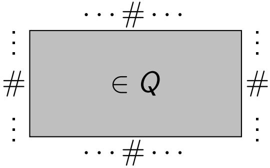

2D recognizability:

1. **tiling recognizable** $\stackrel{def}{=}$ generated by some 2×2 finite type constraints
2. **recognizable** $\stackrel{def}{=}$ projection of the above

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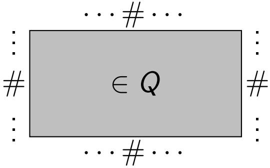

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Detection of $\#$ borders is allowed in tiling recognizability!

Symbolic spaces and subshifts

Setting of this talk:

- *domain:* $D = \mathbb{Z}^2$
- *configurations:* $\mathbb{Z}^2 \to Q$
- *language:* set of finite patterns
- *subshift:* set of configurations avoiding some language
- *subshift of finite type (SFT):* subshift defined by a finite forbidden language
- *sofic subshift:* projection of a SFT

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Subshifts and MSO logic

- formulas always define shift-invariant sets
- formulas don't always define a closed set

Separation / Collapse

Separation

Theorem

There exists a EMSO definable subshift which is not sofic.

- Σ_n^{SO}-defined subshift with Π_n-complete forbidden language

Separation / Collapse

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Theorem

There exists a EMSO definable subshift which is not sofic.

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Collapse at FO level 2 within EMSO

Theorem

Every EMSO-definable set can be defined by a formula of the form:

$$\exists \overline{X}, (\forall \overline{y}, \phi(\overline{y}, \overline{X})) \wedge (\exists \overline{z}, \psi(\overline{z}, \overline{X})),$$

where ϕ and ψ are quantifier-free.

- the uple $\overline{y}$ can always be chosen of size 2
- **proof idea:** threshold counting theorem + technical stuff

SFT and sofic subshifts

Theorem

A set of configurations is an SFT **iff** it can be defined by:

$$\forall z, \phi(z), \text{ where } \phi \text{ is quantifier-free.}$$

- easy proof
- SFT **not** closed by union or complementation

SFT and sofic subshifts

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Theorem

A set of configurations is a sofic subshift **iff** it can be defined by:

$$\exists \overline{X}, \forall \overline{z}, \phi(\overline{X}, \overline{z}), \text{ where } \phi \text{ is quantifier-free.}$$

- more technical proof
- **remark:** such formulas always define closed sets

'Symbolic' characterisation of EMSO

The problem

- sofic subshifts fail to capture all EMSO
- it's not a finite/infinite problem but a **uniformity problem**
- pictures use a # border
- what is really needed?

'Symbolic' characterisation of EMSO

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Breaking uniformity

- fix $Q_0, Q_1 \subseteq Q$ and let C be a Q-configuration
- C is (Q_0, Q_1)-marked $\stackrel{def}{=} \exists z_0, z_1 \; C(z_0) \in Q_0$ and $C(z_1) \in Q_1$
- doubly-marked set of finite type $\stackrel{def}{=}$ set of configurations of a SFT which are (Q_0, Q_1)-marked

'Symbolic' characterisation of EMSO

Theorem

A set is EMSO-definable **iff** it is the projection of a doubly-marked set of finite type.

- **Proof idea**:
  1. threshold counting restricted to a finite zone can be done with DMSFT
  2. show that DMSFT are close by union and intersections
- **Remark:** '#' at SW and NE corners of pictures gives a double marking
- This is true in any dimension (not written, but...)

Open problems

Largest 'sofic' logic fragment

- we have that

  $$\exists \overline{X}, \forall \overline{Y}, \forall \overline{z}, \phi(\overline{X}, \overline{Y}, \overline{z}), \text{ where } \phi \text{ is quantifier-free,}$$

  always defines a sofic subshift.
- how far can we go in SO alternation with sofic subshifts?
- is the whole $*$SO-$\forall$FO fragment sofic?

Infinite alternation hierarchy?

- is the SO alternation hierarchy strict?
- strict for subshifts?
- use complexity of forbidden language?