

# **Synchronization and Limit Behaviors in Cellular Automata**

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# Overview of the talk

**1 Cellular Automata & Limit Behaviors**

**2 Possible Limit**

**3 Typical Limit**

# Overview of the talk

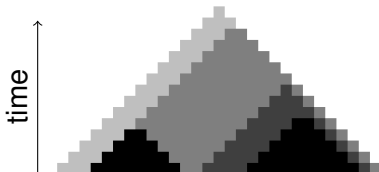
**1 Cellular Automata & Limit Behaviors**

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# One-dimensional CA

- $Q$  set of **states**
- $r$  **radius** of neighborhood
- $f : Q^{2r+1} \rightarrow Q$  **local** transition function
- $Q^{\mathbb{Z}}$  set of **configurations**
- $F : Q^{\mathbb{Z}} \rightarrow Q^{\mathbb{Z}}$  **global** transition function



- $Q = \{0, 1, 2, 3, 4\}$
- $r = 1$
- $f(x, y, z) = \max(x, y, z)$

# Limit behaviors

## Possible

$u$  **limit** word  
 $\Leftrightarrow$   
 $F^{-t}(u)$  **never empty**

 $\Omega$ 

configurations made  
exclusively of  
**limit words**

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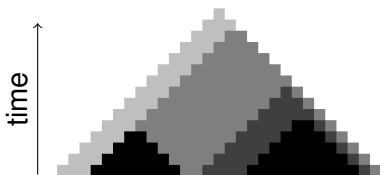
## Typical

$u$   **$\mu$ -limit** word  
 $\Leftrightarrow$   
 $F^{-t}(u)$  **don't get negligible**

 $\Omega_{\mu}$ 

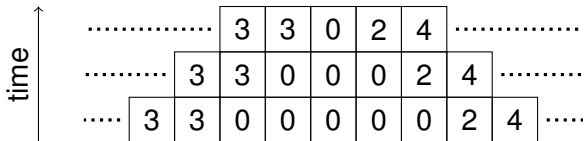
configurations made  
exclusively of  
 **$\mu$ -limit words**

## Limit set ( $\Omega$ )



- $Q = \{0, 1, 2, 3, 4\}$
- $r = 1$
- $f(x, y, z) = \max(x, y, z)$

- $\Omega =$  “*decreasing then increasing*” configurations



## $\mu$ -limit set ( $\Omega_\mu$ )

- $[u]$ : configurations where word  $u$  occurs in the center
- $\mu$  a translation invariant measure (**in this talk: Bernoulli**)

### Definition

- $u$  is a  $\mu$ -limit word if

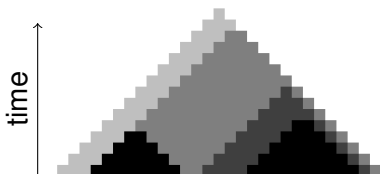
$$\lim_{t \rightarrow \infty} \mu(F^{-t}([u])) \neq 0$$

- $\Omega_\mu$  is the set of configurations made only of  $\mu$ -limit words



**Limit Sets of Cellular Automata  
Associated to Probability Measures**  
*P. Kůrka, A. Maass, 2000*

## $\mu$ -limit set ( $\Omega_\mu$ )



- $Q = \{0, 1, 2, 3, 4\}$
- $r = 1$
- $f(x, y, z) = \max(x, y, z)$

■  $\Omega_\mu = \{\omega 4^\omega\}$

- 1  $u \in (Q \setminus \{4\})^* \Rightarrow$  pre-images of  $u$  in  $(Q \setminus \{4\})^*$
- 2  $\mu((Q \setminus \{4\})^n) \rightarrow 0$  when  $n \rightarrow \infty$



## $\Omega_\mu$ and density

- density of word  $u$  in configuration  $c$

$$d_c(u) = \limsup_{n \rightarrow \infty} \frac{|c_{-n,n}|_u}{2n+1}$$

- configuration  $c$  is  $\mu$ -generic if  $d_c(u) = \mu([u])$  for all  $u$

### Property

The following are equivalent:

- 1  $u$  is a  $\mu$ -limit word for  $F$
- 2 for any  $\mu$ -generic configuration  $c$

$$d_{F^t(c)}(u) \not\rightarrow 0$$

when  $t \rightarrow \infty$

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# Synchronization task #1

**Find some  $F$  such that...**

for all  $t$  there is an initial configuration  $c_t$  with

- 1** all cells are in state 0 at time  $t$
- 2** no 0 appears before time  $t$

# Synchronization task #1

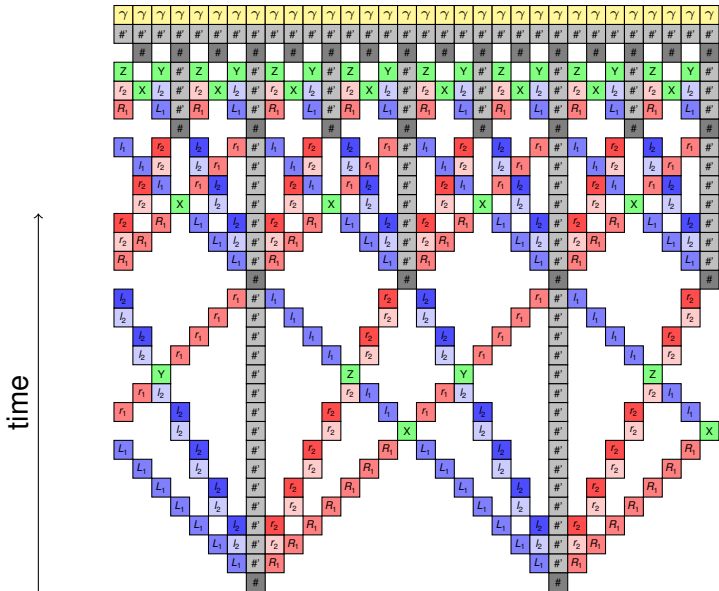
**Find some  $F$  such that...**

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Well known solutions: firing squad CA

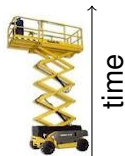
# J. Kari's firing squad



# Applications to $\Omega$

## Firing Squad Elevator

*Raises any configuration to the limit set*



- fix some  $F$  over states  $Q$
- by adding a firing squad component to  $F$ , we can
  - 1 make any word in  $Q^*$  a limit word
  - 2 without changing the dynamics of  $F$  over  $Q^{\mathbb{Z}}$

**Formally:**  $G$  over  $(Q' \times Q) \cup Q$  such that

- 1 the whole set  $Q^{\mathbb{Z}}$  is in  $\Omega(G)$
- 2  $G$  restricted to  $Q^{\mathbb{Z}}$  is exactly  $F$

# Applications to $\Omega$

## Theorem (J. Kari, 1994)

*Any non-trivial property of limit sets is undecidable*

## Theorem (P. Guillon, P.E. Meunier, GT, 2010)

*There is an intrinsically universal CA with a simple limit set*

*(simple = logspace computable)*

# Rice Theorem for $\Omega$

Firing Squad Elevator  
+ Switch



## Definition

$F$  **nilpotent** if  $\Omega(F)$  is a singleton

**Construction:**  $F, H \rightarrow G$

*Is  $H$  nilpotent?*

- **YES:**  $\Omega(G) = \Omega(F)$
- **NO:**  $\Omega(G) = \Omega_0$  independent of  $F$

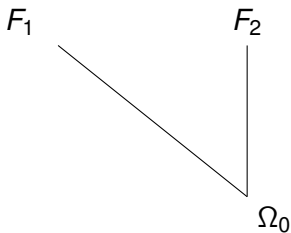


# Rice Theorem for $\Omega$

J. Kari, 1992

Nilpotency is an undecidable property

- fix some property  $\mathcal{P}$  of limit sets
- choose  $F_1$  and  $F_2$  with
  - $\Omega(F_1) \in \mathcal{P}$
  - $\Omega(F_2) \notin \mathcal{P}$
- apply construction twice with the same  $H$



**$H$  not nilpotent**

# Rice Theorem for $\Omega$

J. Kari, 1992

Nilpotency is an undecidable property

- fix some property  $\mathcal{P}$  of limit sets
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- apply construction twice with the same  $H$

$F_1$   
|  
 $\Omega(F_1)$

$F_2$   
|  
 $\Omega(F_2)$

$H$  nilpotent

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## Synchronization task #2

- fix some  $n \geq 2$

**Find some  $F$  such that...**

for **almost all** initial configuration  $c$

- any cell, after some time, is in state  $t \bmod n$  at time  $t$

## Synchronization task #2

- fix some  $n \geq 2$

Find some  $F$  such that...

for **almost all** initial configuration  $c$

- any cell, after some time, is in state  $t \bmod n$  at time  $t$

**A solution exists!**

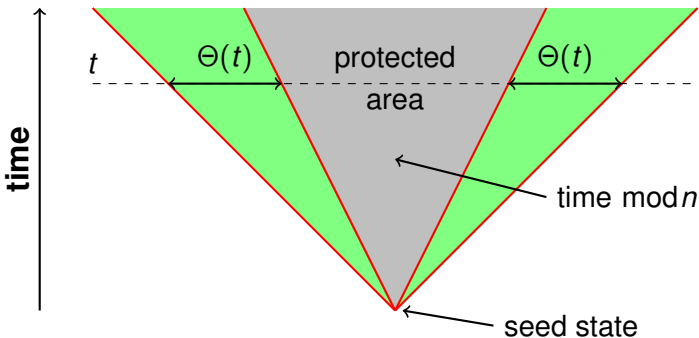


**Directional Dynamics along Arbitrary  
Curves in Cellular Automata**

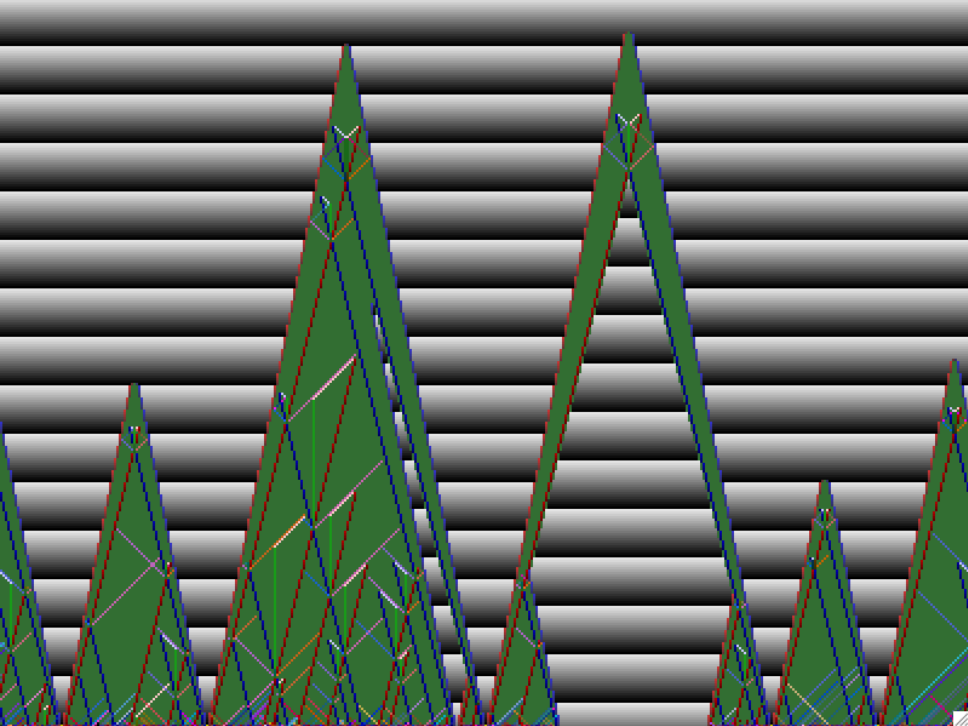
*M. Delacourt, V. Poupet, M. Sablik, GT, 2010*

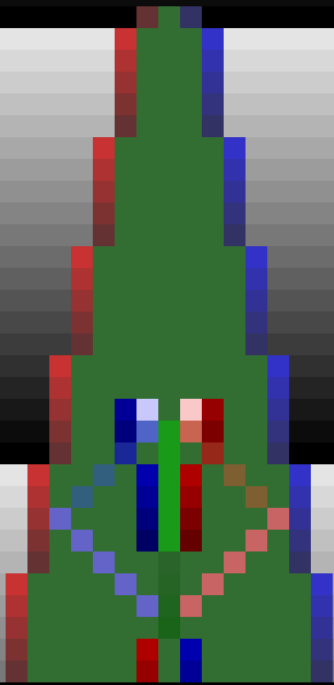
# Time Counters Construction

## Outline

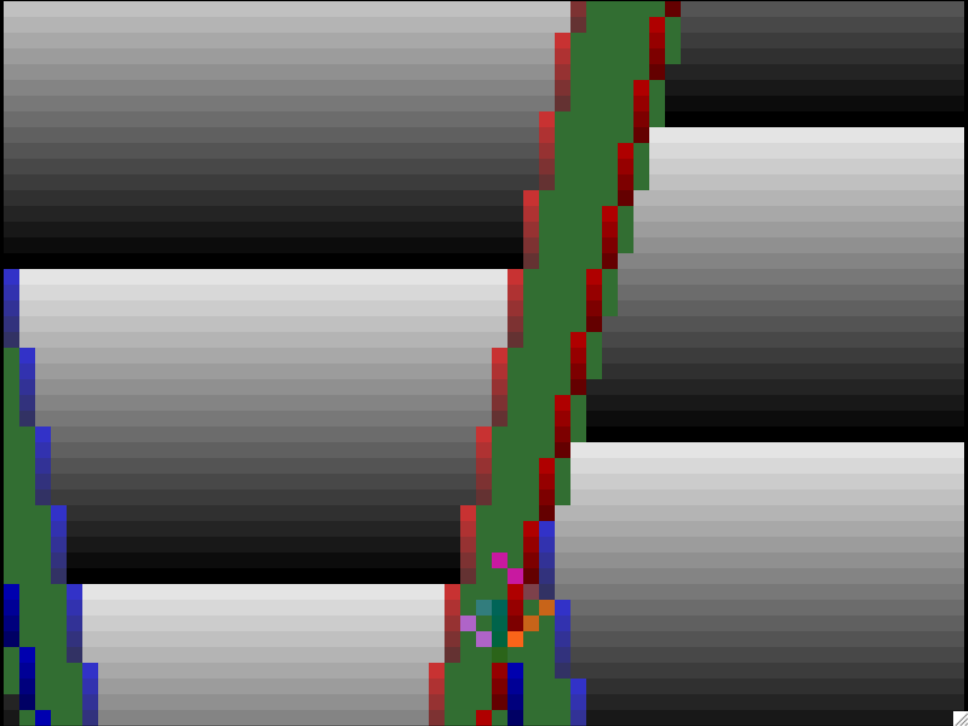


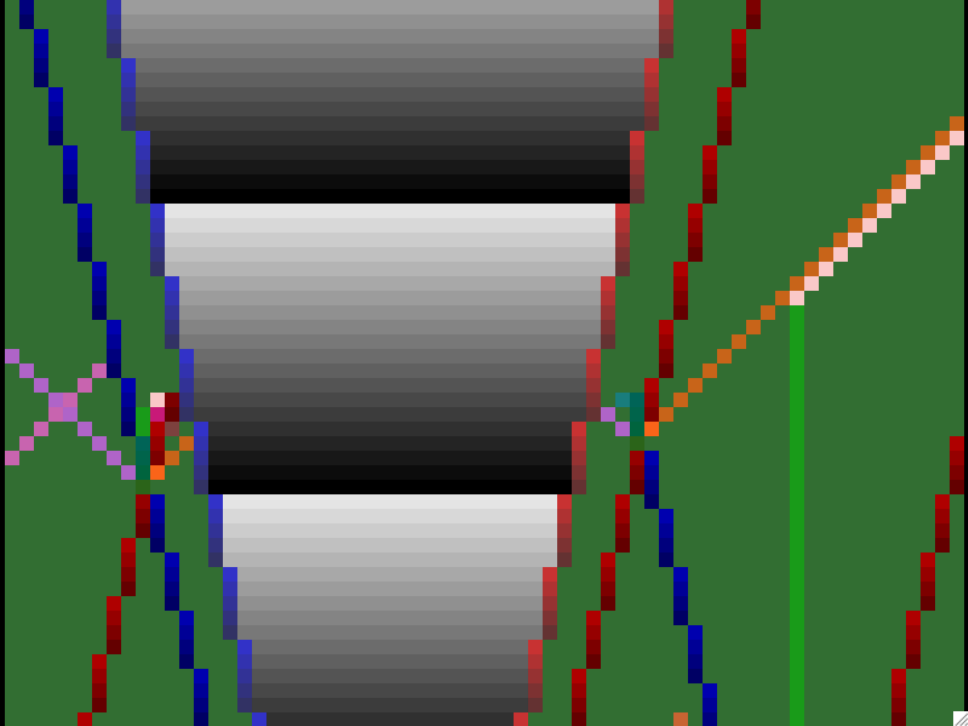
- 1 only a valid zone can stop a valid zone
- 2 when two valid zones meet, the older is destroyed
- 3 two valid zones of equal age merge when they meet











# Time Counters Construction

## Implementation details

Construction for  $n=20$ :

- 2733 states
- radius 4

### Question

Is there a significantly smaller solution?

- Kari's firing squad: 16 states, radius 1
- Mazoyer's firing squad: 6 states, radius 1

# Time Counters Construction

## Implementation details

Construction for  $n=20$ :

- 2733 states
- radius 4

### Question

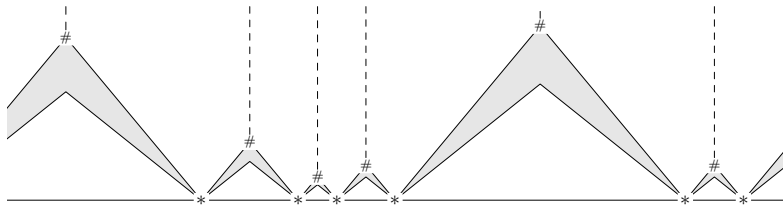
Is there a significantly smaller solution?

- Kari's firing squad: 16 states, radius 1
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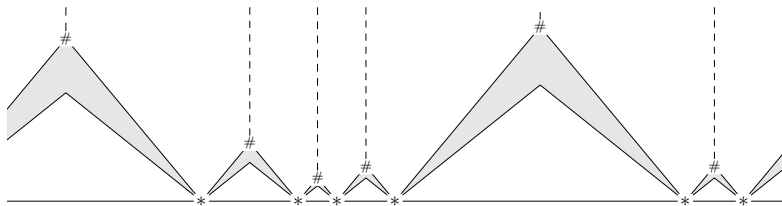
### Other property

CA with equicontinuous points but none in the image set  $F(Q^{\mathbb{Z}})$

## Applications to $\Omega_\mu$






## Applications to $\Omega_\mu$



*Computation segment:*



-  = **computation area** (Turing head + working space)
-  = **merging process info** (time, length, random bits,...)
-  = write once **output**

## Applications to $\Omega_\mu$



**IF**

- segment size  $\rightarrow \infty$
- non-output part  $\ll$  segment size

**THEN**

**Characterization of  $\Omega_\mu$**

$\mu$ -limit word **are exactly** words which are dense in the computation output (asymptotically)

# Applications to $\Omega_\mu$



## **Construction of $\mu$ -limit sets**

*L. Boyer, M. Delacourt, M. Sablik (2010)*



## **Constructions with an ergodic point of view**

*M. Sablik*

*(Information & Randomness 2010, ALEA 2011)*



## **Rice Theorem for $\mu$ -Limit Sets of Cellular Automata**

*M. Delacourt (2011)*



# Rice Theorem for $\Omega_\mu$

## Definition

$F$   **$\mu$ -nilpotent** if  $\Omega_\mu(F)$  is a singleton

- a state is **persistent** if it cannot disappear from a cell

## L. Boyer, V. Poupet, GT, 2006

- $\mu$ -nilpotency is undecidable for CA with a persistent state
- $\mu$ -limit words are enumerable for such CA

**Construction:**  $F, H \rightarrow G$

*Is  $H$   $\mu$ -nilpotent?*

- **YES:**  $\Omega_\mu(G) = \Omega_\mu(F)$
- **NO:**  $\Omega_\mu(G) = \{\mathbb{Z}q\mathbb{Z}\}$  independent of  $F$

## Work in Progress / Future Work

- complex  $\mu$ -limit sets
- higher complexity lower bounds for properties of limit sets
- convergence behaviors (e.g. limit vs. cesaro mean)
- higher dimensions