How Common is Universality in Cellular Automata?

Seminario de Matemáticas Discretas
DIM/CMM

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Outline of the talk

1. Universality in cellular automata

2. 0-1 laws for families with 'local symmetry'

3. Existence results in those families

- **Syntactical object**
  - a state set Q
  - a lattice of cells ($\mathbb{Z}$ in this talk)
  - a neighbourhood $V = \{v_1, \ldots, v_k\}$ (finite subset of $\mathbb{Z}$)
  - local rule $f : Q^k \to Q$

Cellular automata

Definition

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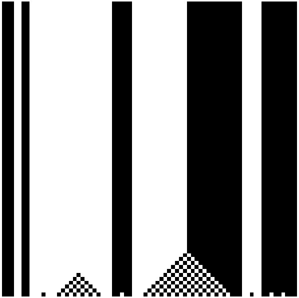
- **Dynamical system**
  - configuration space: $Q^{\mathbb{Z}}$
  - global rule $F : Q^{\mathbb{Z}} \to Q^{\mathbb{Z}}$ defined by:

$$F(x)_z = f\left(x_{z+v_1}, \ldots, x_{z+v_k}\right)$$

Cellular automata

Examples



- $Q = \{0, 1\}$
- $V = \{-1, 0, 1\}$
- $f =$ majority among neighbours

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- $V = \{-1, 0, 1\}$
- $f(x, y, z) = x + y + z \bmod 2$

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  - 4 classes (why 4?)
  - fuzzy definitions (no place for proofs)
  - based on a small set of automata

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- **dynamical systems approach**

  - good for 'chaotic' CA (for others?)
  - which topology?
  - classical notions adapted for CA?

Bulking theory

- **Key ideas**
  - notion of scale
  - a comparison relation rather than a finite number of classes
  - simple formalisation
- **What we get**
  - classification tool (pre-order)
  - nice equivalence relation
  - notion of universality
- **History**
  - J. Mazoyer and I. Rapaport (1998)
  - B. Martin II (2001)
  - N. Ollinger (2002) *(first version with universality)*
  - GT (2005)

Local comparison

- Sub-automaton : $g \sqsubseteq f$

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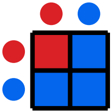
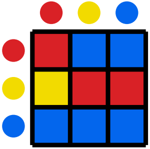


g

$\sqsubseteq$



f

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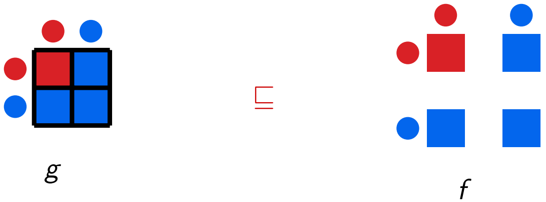

restriction to a stable subset of states

Rescaling

- **3 parameters**

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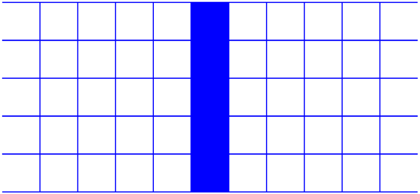


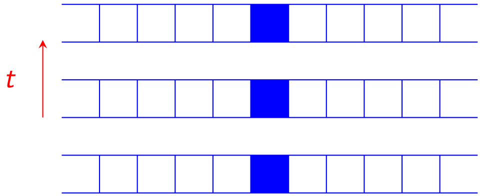
Formally

$$F^{<1,1,0>} = F$$

Rescaling

- **3 parameters**

  - time cutting

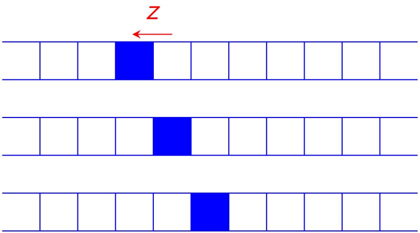


Formally

$$F^{<1,t,0>} = F^{t}$$

Rescaling

- **3 parameters**



- time cutting
- space shifting

Formally

$$F^{<1,t,z>} = \sigma_z \circ F^t$$

Rescaling

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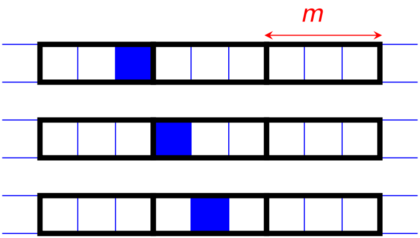


- time cutting
- space shifting
- bulking of cells

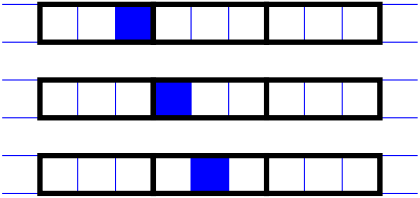
Formally

$$F^{<m,t,z>} = \mathbf{b}_m \circ \sigma_z \circ F^t \circ \mathbf{b}^{-1}{}_m$$

where $\mathbf{b}_m : Q^{\mathbb{Z}} \to (Q^m)^{\mathbb{Z}}$ is the canonical bijection

Rescaling

- **3 parameters**



- time cutting
- space shifting
- bulking of cells

Proposition

$F^{<m,t,z>} : (Q^m)^{\mathbb{Z}} \to (Q^m)^{\mathbb{Z}}$ is always the global function of a cellular automaton.

Bulking and universality

- **Simulation:** sub-automaton relation up to rescaling

Definition

$G \preceq F \iff \exists P_1, P_2 : G^{<P_1>} \sqsubseteq F^{<P_2>}$

- $(\mathrm{CA}, \preceq)$ is a pre-order.

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- **Intrinsic universality:**

Definition

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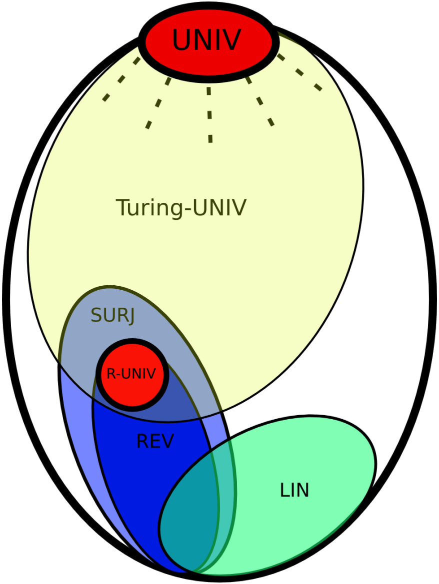
  - intrinsic universality implies 'Turing-universality'
  - (converse **false**)
  - intrinsic universality is undecidable (Ollinger '03)

Bulking and universality



Universality in CA

- **First existence proofs:**
  - 70's: Banks, Conway (Turing-universality)
  - 90's: Culick, Martin (towards intrinsic universality)
  - 00's: Ollinger, Richard (intrinsic universality)

- **How common is universality in CA?**
  - question first raised by Conway (late 70's), then Wolfram (84)
  - we choose the strongest notion: intrinsic universality

Probabilistic Framework

- $\mathcal{P}$ a property (a set of CA)

- $\mathcal{F}$ a family (another set of CA)
- $\mathcal{F}_{n,k}$: CA of $\mathcal{F}$ with n states and k neighbours

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- $\mathcal{F}$ a family (another set of CA)
- $\mathcal{F}_{n,k}$: CA of $\mathcal{F}$ with n states and k neighbours

- Probability of $\mathcal{P}$ in $\mathcal{F}_{n,k}$:

$$Pr\left(\mathcal{P}|\mathcal{F}_{n,k}\right) = \frac{\#\left(\mathcal{F}_{n,k} \cap \mathcal{P}\right)}{\#\mathcal{F}_{n,k}}$$

- Probability of $\mathcal{P}$ in $\mathcal{F}$?

Probabilistic Framework

- **Asymptotic density**
  - **idea:** the limit of $Pr\left(\mathcal{P}|\mathcal{F}_{n,k}\right)$ when 'size'$\rightarrow \infty$
  - problem: 2 parameters for 'size' (n and k)

Probabilistic Framework

- **Asymptotic density**
  - **idea:** the limit of $Pr\big(\mathcal{P}|\mathcal{F}_{n,k}\big)$ when 'size'$\rightarrow \infty$
  - problem: 2 parameters for 'size' (n and k)

- **Asymptotic density along a path**
  - a path is an injective map $p : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$
  - equivalently: a sequence of pairs (n, k) without repetition
  - the density along p is:

$$d_{p,\mathcal{F}}(\mathcal{P}) = \lim_{t\rightarrow\infty} Pr\big(\mathcal{P}|\mathcal{F}_{p(t)}\big)$$

0-1 laws

Definition (Increasing properties)

$\mathcal{P}$ increasing in $\mathcal{F}$ when for all $F, G \in \mathcal{F}$:

$$\text{if } F \preceq G \text{ then } F \in \mathcal{P} \Rightarrow G \in \mathcal{P}$$

Remark: intrinsic universality is an increasing property!

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Remark: intrinsic universality is an increasing property!

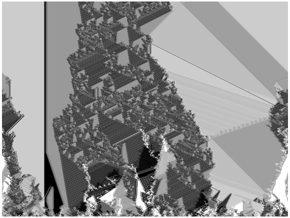
- **0-1 laws for monotone properties**

Definition

$\mathcal{F}$ has the 0-1 law for path p if for all **non-trivial** *$\mathcal{P}$:*

- *$\mathcal{P}$ increasing* $\Rightarrow d_{p,\mathcal{F}}(\mathcal{P}) = 1$
- *$\mathcal{P}$ decreasing* $\Rightarrow d_{p,\mathcal{F}}(\mathcal{P}) = 0$

Captive CA



- **local constraint**

$$f(x_1, \ldots, x_k) \in \{x_1, \ldots, x_k\}$$

Proposition

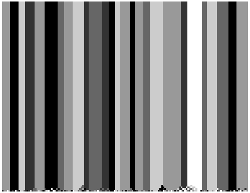
0-1 law for paths with **fixed neighbourhood**

Majority CA

▶ **local constraint**

$f(x_1, \ldots, x_k) \in \{x_i : \#(i) \geq \#(j), \forall j\}$

where $\#(i) = \#\{j : x_j = x_i\}$



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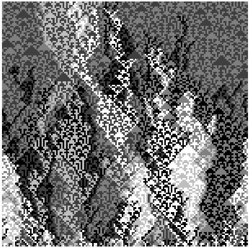


Proposition

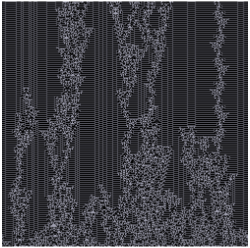
0-1 law for paths with **fixed neighbourhood**

Multi-set CA

- **local constraint**

$$\forall \pi \in \mathfrak{S}_k :$$

$$f(x_1, \ldots, x_k) = f(x_{\pi(1)}, \ldots, x_{\pi(k)}).$$



Proposition

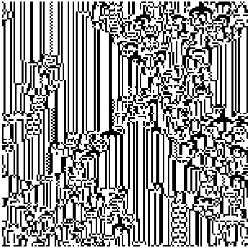
0-1 law for paths with **fixed state set**

Outer-totalistic CA

- **local constraint**

If $\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i$ and $x_m = y_m$

then $f(x_1, \ldots, x_k) = f(y_1, \ldots, y_k)$.



Proposition

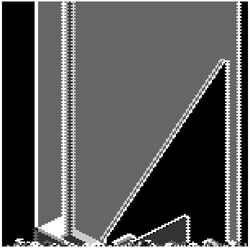
0-1 law for paths with **fixed state set**

Set + captive CA

- **local constraint of 'set CA':**

If $\{x_1, \ldots, x_k\} = \{y_1, \ldots, y_k\}$,

then $f(x_1, \ldots, x_k) = f(y_1, \ldots, y_k)$.



Proposition

0-1 law for any path with **states**$\rightarrow \infty$

Summary

Family $\mathcal{F}$	**Path condition**
Captive	fixed neighb.
Minority	fixed neighb.
Majority	fixed neighb.
Multiset	fixed states
Outer-Totalistic	fixed states
Set + captive	any p with $n \to \infty$
Multiset + captive	any p with $n \to \infty$

Some words about proofs

- **Key property**
  - $F_0 \in \mathcal{F}$ fixed
  - $\mathcal{P}_0 = \{F : F_0 \preceq F\}$
  - we prove: $d_{p,\mathcal{F}}(\mathcal{P}_0) = 1$

Some words about proofs

- **Key property**
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- **Proof scheme**
  1. in a large F, find many 'ways' (w_i) for F to simulate F_0
  2. prove that events '$F_0 \preceq F$ in the way w_i' are independent
  3. lower bound probability of those events uniformly

- *very different arguments for step 1 depending on* $\mathcal{F}$

Last step...

- **What we have so far:**
  - 0-1 laws for various $\mathcal{F}$ (with path conditions)
  - consequence for universality in $\mathcal{F}$:

    'if there is 1 universal CA in $\mathcal{F}$ then almost all are universal'

Last step...

- **What we have so far:**
  - 0-1 laws for various $\mathcal{F}$ (with path conditions)
  - consequence for universality in $\mathcal{F}$:

    '*if there is 1 universal CA in $\mathcal{F}$ then almost all are universal*'

- **Is there a single universal CA in $\mathcal{F}$?**
  - non-trivial!
  - local constraints may avoid universality (*e.g.* linearity)
  - we will show more than existence...

Encodings

- **Definitions:**
  - $\phi : \mathsf{CA} \rightarrow \mathcal{F}$ is an encoding for $\mathcal{F}$ if
    1. it is computable and injective
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- **Remarks**
  - encoding for $\mathcal{F} \Rightarrow \mathcal{F} \cap \mathbf{Univ} \neq \emptyset$
  - faithful encoding for $\mathcal{F} \Rightarrow$ **Univ** undecidable in $\mathcal{F}$

Results

Family $\mathcal{F}$	**Encoding**	**Faithful encoding**
Captive	yes	yes
Minority	yes	?
Majority	yes	?
Multiset	yes	yes
Outer-Totalistic	yes	yes
Set + captive	yes	?
Multiset + captive	yes	?

- *technical proofs (especially faithful encodings)*
- *no general method*

Unresolved problems

- prove faithful encoding for each family
- most general path conditions in each family?
- a unified explanation behind all those 'symmetric' families
- density of universality in the whole set of CA?

Future directions

- extension to other 'natural' families (e.g. number conserving)
- extension to higher dimension / other lattices of cell
- study other properties than universality
- most general family with density 0 for **Univ**?