Don't Forget the Lattice!

Universidad de Concepción

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November 7th 2012

Overview of the talk

1. What it is all about
2. A surprising characterization
3. Bounded Time Properties
4. Topological Dynamics

Overview of the talk

Symbolic spaces

$$Q^L$$

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- **finite set** (alphabet or states or colors...)

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$$Q^L$$

- **the “lattice”**
  - a monoid or a group
  - law denoted '$+$'
  - finitely generated
  - *typically:* $\mathbb{Z}^d$

Pro-discrete topology

- configuration

$$c : L \to Q$$

- finite pattern

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Q^L is compact

Cellular automata

- *Syntactical object (given)*
  - **neighborhood**: a finite domain D
  - **local rule**: $f : Q^D \to Q$

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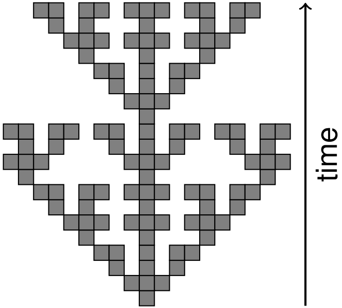
  - **neighborhood**: a finite domain D
  - **local rule**: $f : Q^D \rightarrow Q$
- *Dynamical system (studied)*
  - global function : $F : Q^L \rightarrow Q^L$ s.t.

$$F(c)_z = f(c_{[D,z]})$$

where $c_{[D,z]}$ is the finite pattern : $z' \in D \mapsto c(z + z')$

Example: local sum mod 2

- $L = \mathbb{Z}$
- $Q = \{0, 1\}$
- $D = \{-1, 0, 1\}$
- $f(x, y, z) = x + y + z \bmod 2$



Topological characterization

- action of L on configurations: shift σ_z

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F is a cellular automaton **iff** it is continuous and shift invariant.

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F is a cellular automaton **iff** it is continuous and shift invariant.

Corollary: if a CA is bijective then its inverse is also a CA

Don't forget the lattice!

- **Temptation:** study CA through topological abstraction

$$(K, F)$$

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- This talk:

  1. interesting properties which holds only for some lattices
  2. topological abstraction hides complexity issues
  3. are we really doing topology/analysis or just combinatorics/arithmetics?

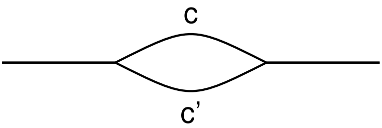
Overview of the talk

Garden of Eden Theorem

- c and c' **finitely different**, $c \neq_f c'$, if

$$c_{|D} \neq c'_{|D} \text{ but } c_{|D^c} = c'_{|D^c}$$

  for some finite domain $D \neq \emptyset$



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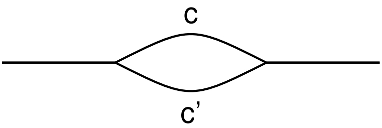

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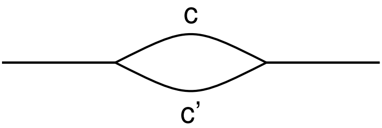

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Moore-Myhill Theorem ($L = \mathbb{Z}^2$)

F is pre-injective **iff** it is surjective

Examples

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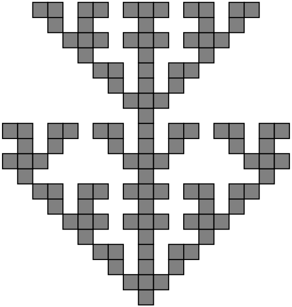

- $L = \mathbb{Z}$
- $Q = \{0, 1\}$
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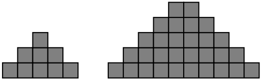
- $f(x, y, z) = xyz$



Amenable group

- let L be a group
- L amenable if $\exists$ some L-invariant measure μ on L:
  1. $\mu(L) = 1$
  2. $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A \cap B = \emptyset$
  3. $\mu(g + A) = \mu(A)$ for all $g \in L$ and $A \subseteq L$

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- counter-example: the free group with 2 generators

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- **examples**:
  - $\exists x, F(x) = x$ (**having a fixed point**)
  - $\forall y, \exists x, F(x) = y$ (**being surjective**)
  - $\forall x, \forall y, F(x) = F(y) \Rightarrow x = y$ (**being injective**)

$L = \mathbb{Z}$, the realm of finite automata

- fix some bounded time property $\mathcal{P}$

- what is the complexity of the following problem:

input	the local definition of a CA F
question	does F verify $\mathcal{P}$?

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- ω-automata and model checking

$L = \mathbb{Z}^2$, the realm of tilings

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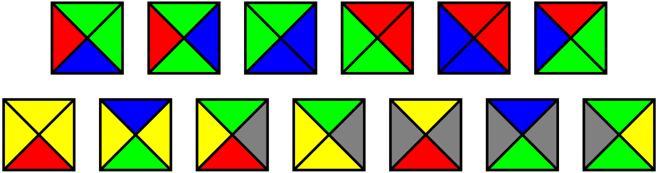
Proof: direct encoding of the domino problem which is undecidable (Berger, 1966)



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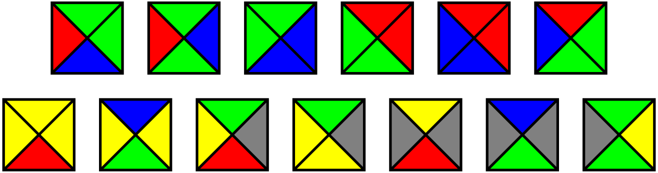

Theorem (Kari, 1990, 1994)

When the lattice is $\mathbb{Z}^2$, both injectivity and surjectivity are undecidable properties.

Higher Undecidability?

integers $\equiv$ finite objects $\equiv$ CA $\equiv$ finite words $\equiv \cdots$

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Arithmetical hierarchy: X is *arithmetical at level k* if

$$n \in X \Leftrightarrow \underbrace{\forall n_1 \exists n_2 \cdots}_{k \text{ alternations}} \underbrace{R(n, n_1, n_2, \ldots)}_{\text{computable relation}}$$

where n and n_i are all integers.

level 0 $\equiv$ decidable

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Theorem (unpublished)

Any bounded time property is arthimetical at some level independent of the lattice.

Some Questions

- how arithmetical level of some property can change with lattice?
- a property at different levels for $\mathbb{Z}^2$ and $\mathbb{Z}^3$?
- which properties are decidable for any lattice?

Open problem (Gottschalk, 1973)

Is there a group with an injective CA which is not surjective?

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Topological Dynamics

- **intuition:** using distance + unbounded time

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- **language**:
  1. **objects:** configurations ($\mathbf{x}$, $\mathbf{y}$,...) and integers (t, n,...)
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  3. quantification over configurations and integers.
- **Eq** $\equiv$ **having an equicontinuous point**

$$\exists \mathbf{x}, \forall n, \exists m, \forall \mathbf{y}, \forall t,\ d(\mathbf{x}, \mathbf{y}) \leq \frac{1}{m} \Rightarrow d\big(F^t(\mathbf{x}), F^t(\mathbf{y})\big) \leq \frac{1}{n}$$

- **S** $\equiv$ **sensitive to initial conditions**

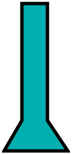
$$\exists n, \forall \mathbf{x}, \forall m, \exists \mathbf{y}, \exists t,\ d(\mathbf{x}, \mathbf{y}) \leq \frac{1}{m} \text{ and } d\big(F^t(\mathbf{x}), F^t(\mathbf{y})\big) > \frac{1}{n}$$

- **X** $\equiv$ **positively expansive**

$$\exists n, \forall \mathbf{x}, \forall \mathbf{y}, \exists t,\ \mathbf{x} \neq \mathbf{y} \Rightarrow d\big(F^t(\mathbf{x}), F^t(\mathbf{y})\big) > \frac{1}{n}$$

$L = \mathbb{Z}$, the realm of information walls

- F a CA with radius r
  - Obstacle



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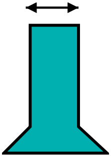

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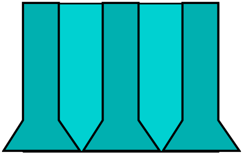

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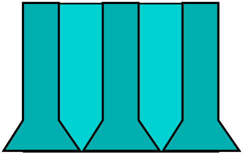

Theorem (Kůrka, 1997)

1. $F \in \mathbf{S} \iff F$ has no wall $\iff F \notin \mathbf{Eq}$
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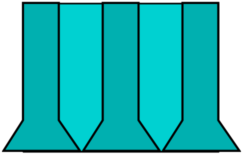

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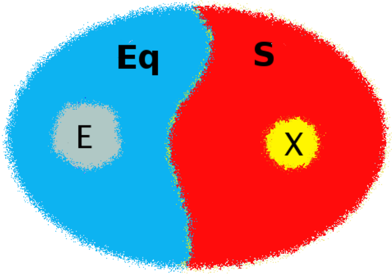
- **intuition:** equicontinuous point = infinite concatenation of walls...

$L = \mathbb{Z}$, the realm of information walls

- P. Kůrka's dynamical classification for **1D** CA:



- additional properties:
  - sensitivity constant computable for $F \in \mathbf{S}$
  - $X \neq \emptyset$

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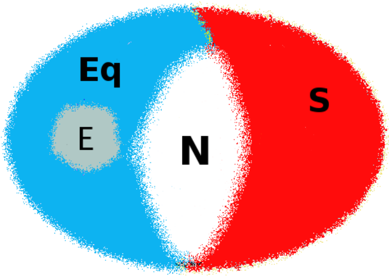
Theorem (Sablik-Theyssier, 2008)

1. there exists F outside $\mathbf{S} \cup \mathbf{Eq}$
2. sensitivity constant is uncomputable for $F \in \mathbf{S}$
3. $\exists F \in \mathbf{Eq}$ having only non-recursive equicontinuity points



Higher undecidability

Theorem (Sablik-Theyssier, 2009)

The set **S** of sensitive CA is

- arithmetical at most at level 2 for $L = \mathbb{Z}$
- at least at level 3 for $L = \mathbb{Z}^3$

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Theorem (unpublished)

The set **S** is arithmetical and at most at level 3 for any lattice.

Some Questions (top. dyn.)

- what about arithmetical level of **S** for $\mathbb{Z}$ and $\mathbb{Z}^2$?
- a property with unbounded level when lattice changes?
- a topological property which is not arithmetic?
- variants of expansivity for $\mathbb{Z}^d$ with $d \geq 2$?


¡Gracias!