Topological Dynamics of 2D Cellular Automata

CiE 2008, Athens

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Overview of the talk

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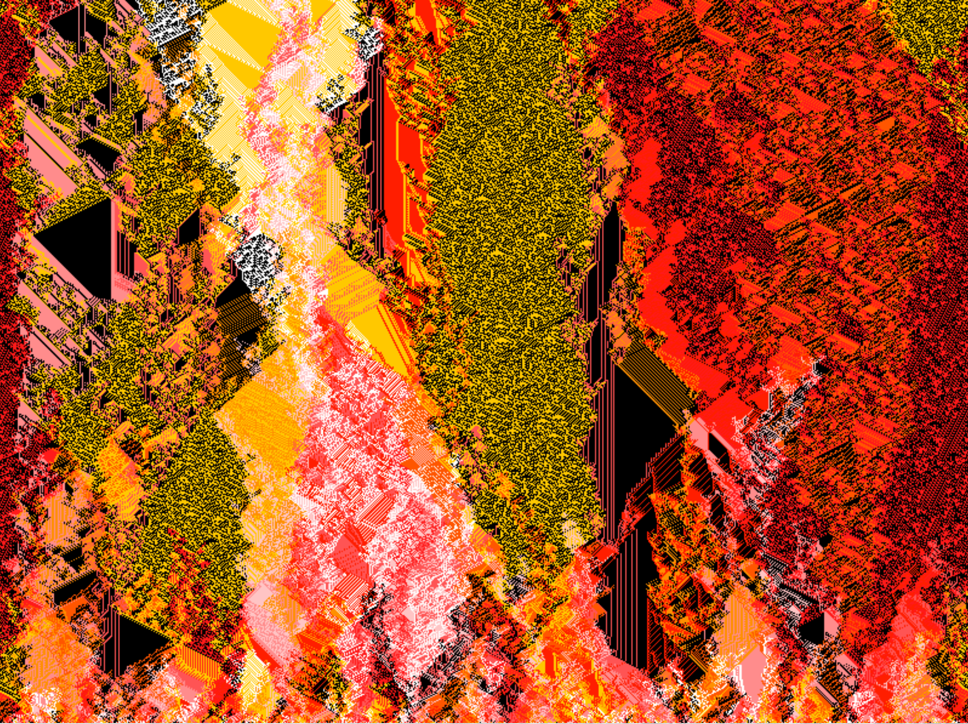
Definition

- **Syntactical object:**
  - Q state set,
  - d dimension,
  - $V = \{v_1, \ldots, v_k\} \subseteq \mathbb{Z}^d$ neighborhood,
  - $\delta : Q^V \to Q$ local transition rule
- **Associated behavior:**
  - $Q^{\mathbb{Z}^d}$: set of configurations
  - $F_A : Q^{\mathbb{Z}^d} \to Q^{\mathbb{Z}^d}$ defined by

$$F_A(x)_z = \delta\big(x_{z+v_1}, \ldots, x_{z+v_k}\big)$$

- **Example**



CA as Dynamical Systems

- **Cantor distance:**
  - $D(x, y)$= dist. to center of the 1st cell where x and y differ
  - Cantor distance: $d(x, y) = 2^{-D(x,y)}$
  - the space of configurations is compact

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Theorem (Curtis, Hedlund, Lyndon, 69)

CA global functions are exactly the continuous function which commute with shifts.

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Theorem (Curtis, Hedlund, Lyndon, 69)

CA global functions are exactly the continuous function which commute with shifts.

- CA fit into the formalism of dynamical systems theory...
- let's study them through this lens!

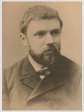
Overview of the talk

1. Cellular Automata
2. **Topological Dynamics**
3. The Core Construction
4. The Onion Skin Trick
5. Research Directions

Deterministic Chaos
On the Flap of Butterflies' Wings

"*Small differences in initial conditions may induce large differences in final phenomena.*"

H. Poincaré. *Calcul des probabilités* (Paris, 1912)



Sensitivity to initial conditions:

$$\exists\epsilon, \forall x, \forall\delta, \exists y, \exists t : d(x, y) \leq \delta \text{ and } d(F^t(x), F^t(y)) \geq \epsilon$$

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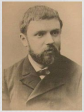

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Equicontinuity point at *x*:

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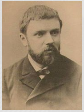

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- **S**= set of sensitive CA
- **Eq**= set of CA having equicontinuity points

Equicontinuity and information propagation

Our butterfly can set off a tornado but cannot cross walls...

- F a 1D CA with radius r
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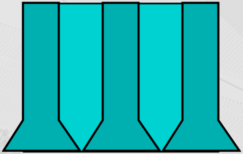

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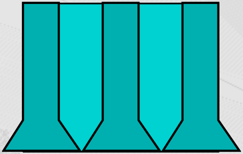

Proposition (P. Kůrka, 1997)

$F \in \boldsymbol{S} \iff F \textit{ has no wall} \iff F \notin \boldsymbol{Eq}$

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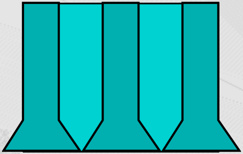


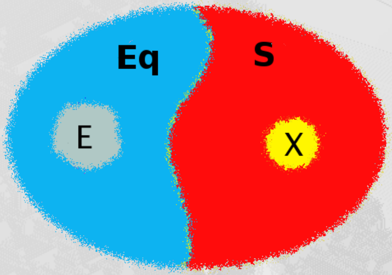
Proposition (P. Kůrka, 1997)

$F \in \boldsymbol{S} \iff F$ *has no wall* $\iff F \notin \boldsymbol{Eq}$

- **intuition:** equicontinuity point = infinite concatenation of walls...

The picture for 1D CA

- P. Kůrka's dynamical classification for **1D** CA:



- additional properties:
  - sensitivity constant computable for $F \in \mathbf{S}$
  - if $F \in \mathbf{Eq}$, it has a periodic equicontinuity point

Topological Dynamics of **2D** CA

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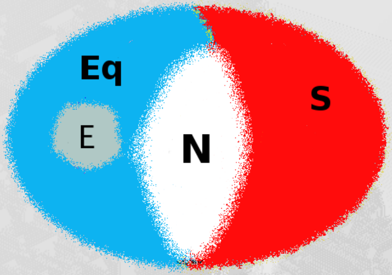
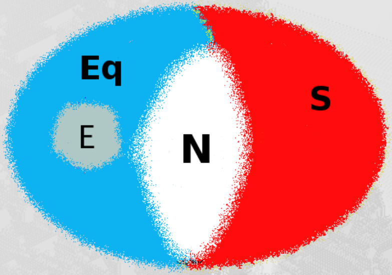
- **A new picture**



Topological Dynamics of **2D** CA

▸ **A new picture**



▸ **Other striking 1D/2D differences**

- sensitivity constant is uncomputable for $F \in$ **S**
- $\exists F \in$ **Eq** having only non-recursive equicontinuity points

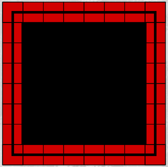
Overview of the talk

Non-sensitive and without equicontinuity point

F is made of 2 components:

Obstacles: *almost static*

- exterior, interior, shell
- local validity check
- on error $\rightarrow$ exterior state



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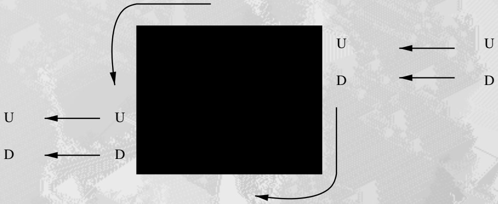
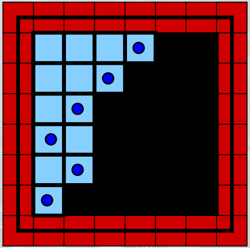
Particles: *simple dynamics*

- made of 2 parts U and D
- U above D = move to the west
- separation to bypass obstacles



Computability results

- **Variations over the previous construction:**
  - put a Turing computation via Wang tiles inside obstacles
  - halting time = upper bound on the size of valid obstacles
  - **other choice**: halting time = lower bound



Computability results

- **Variations over the previous construction:**
  - put a Turing computation via Wang tiles inside obstacles
  - halting time = upper bound on the size of valid obstacles
  - **other choice**: halting time = lower bound
- **Strong undecidability result as a corollary:**

Theorem

*The classes **Eq**, **S** and **N** are neither r.e. nor co-r.e. and they are pairwise recursively inseparable.*

  - uses another construction...

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Building uncomputable onions...

- new definition of obstacles
- succession of “skins” like onions
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Theorem

There exists F having only non-recursive equicontinuity points.

Overview of the talk

Some interesting research directions

- position in the arithmetical hierarchy of Kůrka's classes:
  - in 1D: **Eq** $\in \Sigma_2^0$ and **S** $\in \Pi_2^0$, lower bounds are Π_1^0 and Σ_1^0
  - in 2D: no better lower bounds, upper bounds?

- find and study a notion of (positive) expansivity for higher dimensions

- extend known measure-theoretic properties of **Eq** to **N** (e.g., μ-limit language)

- a direct characterization of **N** through obstacle language?

Questions?