Selfsimilarity, Simulation and Spacetime Symmetries

AUTOMATA 2011

V. Nesme and *G. Theyssier*

Universität Potsdam,
CNRS/Université de Savoie

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Overview of the talk

1. **Simulations, Reversibility, Time Symmetry**
2. **Isolated Dependencies**
3. **Application to Linear Cellular Automata**

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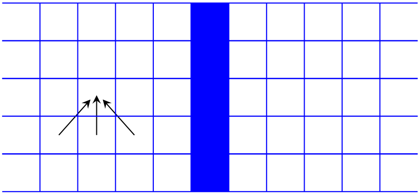
3. Application to Linear Cellular Automata

Space-time rescaling

- **3 parameters:** $F \mapsto F^{\langle m,t,z \rangle}$



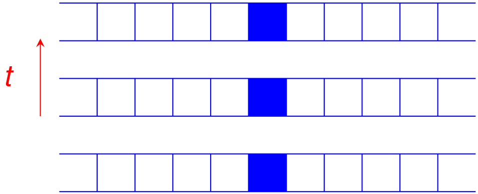
Global map

$$F^{\langle 1,1,0 \rangle} = F$$

Space-time rescaling

- **3 parameters:** $F \mapsto F^{\langle m,t,z \rangle}$

■ time



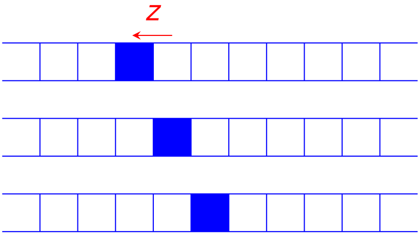
Global map

$$F^{\langle 1,t,0 \rangle} = F^{t}$$

Space-time rescaling

- **3 parameters:** $F \mapsto F^{\langle m,t,z \rangle}$

  - time
  - shift



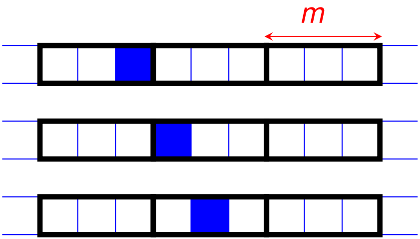
Global map

$$F^{\langle 1,t,z \rangle} = \sigma_z \circ F^t$$

Space-time rescaling

▶ **3 parameters:** $F \mapsto F^{\langle m,t,z \rangle}$

- time
- shift
- cell grouping



Global map

$$F^{\langle m,t,z \rangle} = \mathbf{o}_m^{-1} \circ \sigma_z \circ F^t \circ \mathbf{o}_m$$

$\mathbf{o}_m^{-1} : Q^{\mathbb{Z}} \to (Q^m)^{\mathbb{Z}}$ canonical bijection

Space-time rescaling

▶ **3 parameters:** $F \mapsto F^{\langle m,t,z \rangle}$

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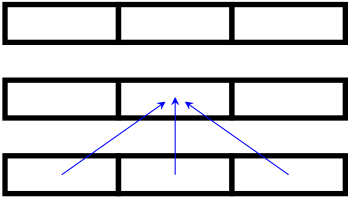


Global map

$F^{\langle m,t,z \rangle}$ is a cellular automaton

- with a possibly different alphabet
- with a possibly different radius

Simulations

Definition

F is a subautomaton of G if (up to renaming of states)

$$F = G_{|X}$$

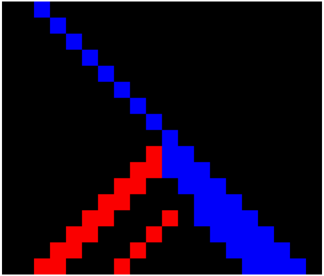
where $X \subseteq Q_G$.

Definition

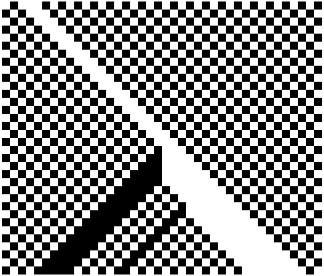
$F \preceq_i G$ *if some rescaling of F is a subautomaton of some rescaling of G*

- $\sim$: equivalence relation induced by $\preceq_i$
- *e.g.* nilpotent CA form a $\sim$-class

Example



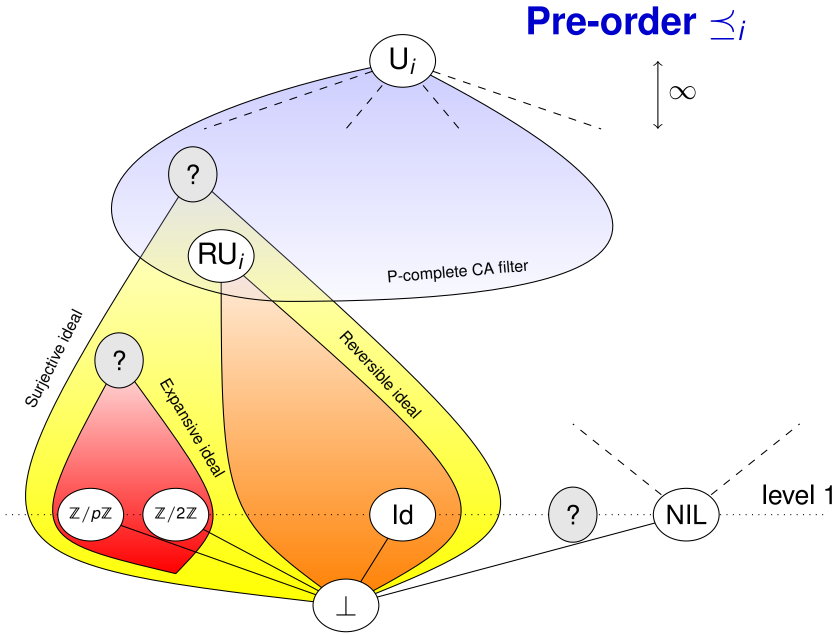
'Just Gliders'



ECA 184

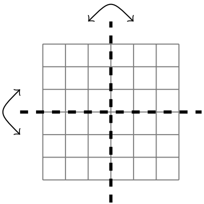
'just Gliders' $\preceq_i$ ECA 184

'Just Gliders' is $184^{\langle 2,2,0 \rangle}$ *restricted to* $\{$■■, □□, ■□$\}$



Pre-order $\preceq_i$
U$_i$
∞
?
RU$_i$
P-complete CA filter
Surjective ideal
?
Expansive ideal
Reversible ideal
level 1
$\mathbb{Z}/p\mathbb{Z}$
$\mathbb{Z}/2\mathbb{Z}$
Id
?
NIL
$\perp$

Reversible CA



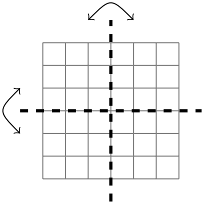
inverse $F \;\rightarrow\; F^{-1}$

mirror $F \;\rightarrow\; M \circ F \circ M$

(where $M(c)_z = c_{-z}$)

dual $F \;\rightarrow\; \tilde{F} = M \circ F^{-1} \circ M$

Reversible CA



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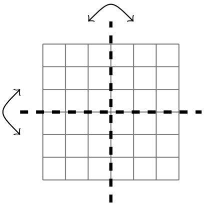
F is self-dual if $F \sim \tilde{F}$

Reversible CA



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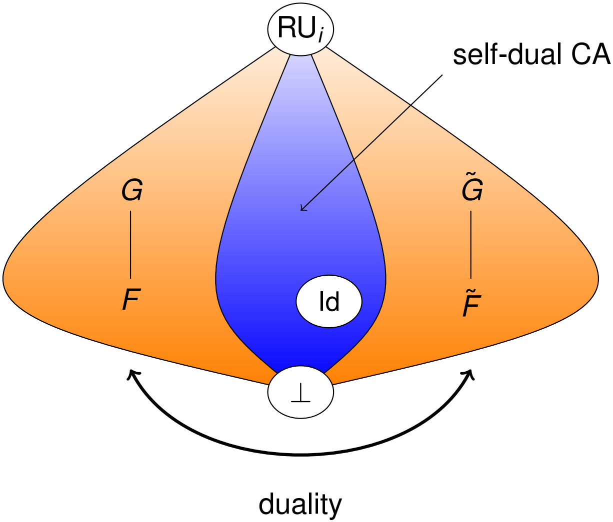
Striking: "hand-made" reversible CA are "often" self-dual

"*Time-Symmetric Cellular Automata*"
Andrés Moreira, Anahi Gajardo (JAC2010)

$\preceq_i$ restricted to reversible CA



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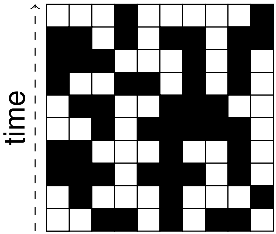
3. Application to Linear Cellular Automata

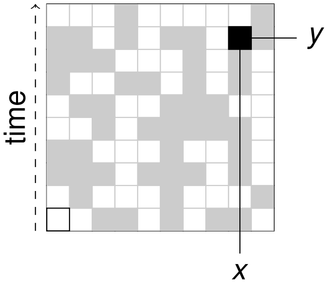
Dependencies

- fix initial configuration c



Dependencies

- fix initial configuration *c*
- observe cell *x* after *y* steps



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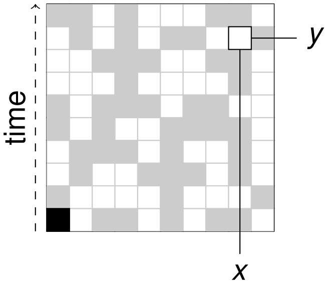

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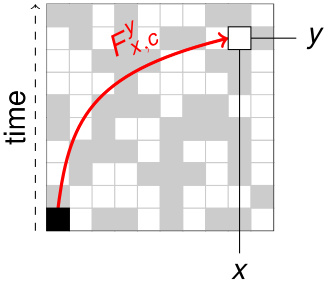

Dependency function

$$F^y_{x,c} : Q \to Q$$

Isolated dependencies

Two extremal cases

- $F_{x,c}^{y}$ bijective (full-dependency)
- $F_{x,c}^{y}$ **constant** (no dependency)

Isolated dependencies

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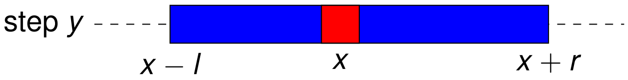
Property Spot[x,y,l,r]

For all configurations c:

- $F^y_{x,c}$ is a bijection;
- $F^y_{z,c}$ is **constant** for all $z \in [x - l; x + r] \setminus \{x\}$



When F is linear

F **linear:**

- $(Q, +, 0)$ a group
- $F(c\overline{+}c') = F(c)\overline{+}F(c')$

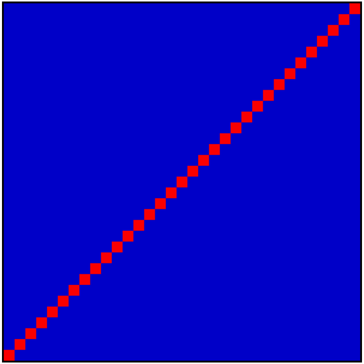
Property

If F is linear then

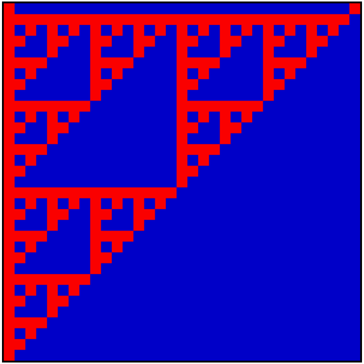
$$F^y_{x,c} \text{ bijective (resp. constant)} \Leftrightarrow F^y_{x,\overline{0}} \text{ bijective (resp. constant)}$$

- **Proof:** $F^y_{x,c}(q) = F^y_{x,\overline{0}}(q) + F^y(c)_x$
- **Consequence:** it is enough to look at a finite number of space-time diagrams to decide property **Spot**

Examples



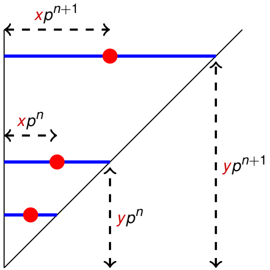
Shift



$x + y \bmod 2$

Scale-free dependency structures

- fix F and $p \geq 2$
- geometric progression of localized dependencies



Definition

$(x, y) \in X_p \subseteq \mathbb{R} \times \mathbb{R}_+$ *iff*

$$Spot[xp^n, yp^n, p^{n-k}, p^{n-k}]$$

for some k and for all large enough n

Scale-free dependency structures

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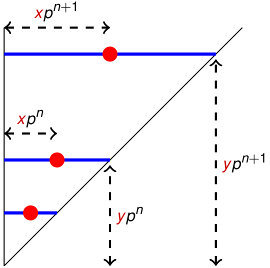

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for some k and for all large enough n

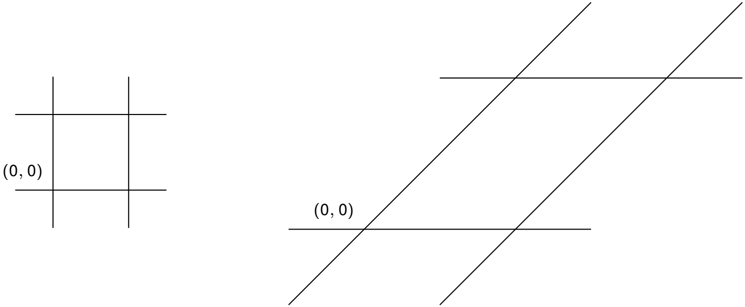
- **Proposition:** $X_p(F) \neq \{(0,0)\} \;\Rightarrow\; F$ surjective

Main Theorem

Theorem

If $F \preceq_i G$ *then* $\phi(\overline{X_p(G)}) \subseteq \overline{X_p(F)}$.

- ϕ transformation of the form $(x, y) \mapsto (\alpha x + \beta y, \gamma y)$ with $\alpha, \gamma > 0$



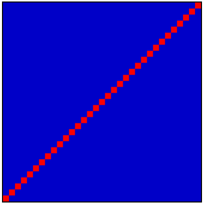
Applications of Main Theorem

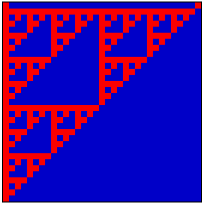
- X_p as an obstacle to simulation:

  the bigger your X_p, the less you can simulate

Examples

- a surjective CA cannot simulate a non-surjective CA
- "$x + y$ mod 2" cannot simulate a shift





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Computing X_p for linear CA

"*The fractal structure of cellular automata on Abelian groups*"
J. Gütschow, V. Nesme, R. F. Werner (AUTOMATA 2010)

Proposition

F linear* $\Rightarrow$ $\left(F_x^y\right)$ can be described by a 2D substitution system

E finite set and $e : \mathbb{Z} \times \mathbb{N} \to E$ with

- e is the fixed-point of a $(p \times p)$-substitution

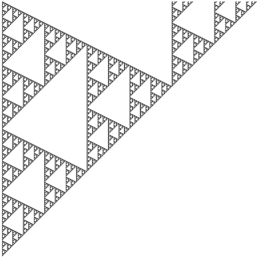
$$\boxed{\alpha} \mapsto \begin{array}{|c|c|}\hline \alpha_{1,1} & \alpha_{2,1} \\ \hline \alpha_{1,2} & \alpha_{2,2} \\ \hline \end{array}$$

- F_x^y is completely determined by $e(x, y)$

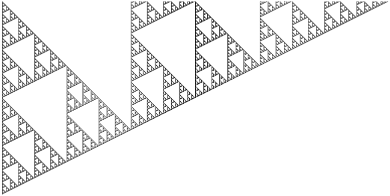
Example: Γ

$$Q = \{0, 1\}^3$$

$$\Gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & X \\ 1 & X & 0 \end{pmatrix}$$



Γ



Γ^{-1}

Example: Γ

Proposition

Γ is not self-dual and cannot simulate the identity

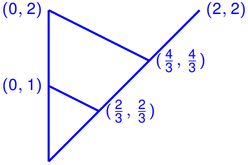
Example: Γ

Proposition

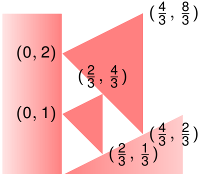
Γ is not self-dual and cannot simulate the identity

Proof

1. 2×2 substitution systems for Γ and Γ^{-1}
2. deduce some properties of $X_2(\Gamma)$ and $X_2(\Gamma^{-1})$



in $X_2(\Gamma)$



not in $X_2(\Gamma^{-1})$

3. conclude with main theorem

Heuristic