

Decidability and Universality in Stochastic Cellular Automata

Rencontres ACP

P. Arrighi, N. Schabanel, **G. Theyssier**

LAMA (CNRS, Université de Savoie, France)

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Overview of the talk

1 Definitions

2 Decision problems

3 Simulations

Overview of the talk

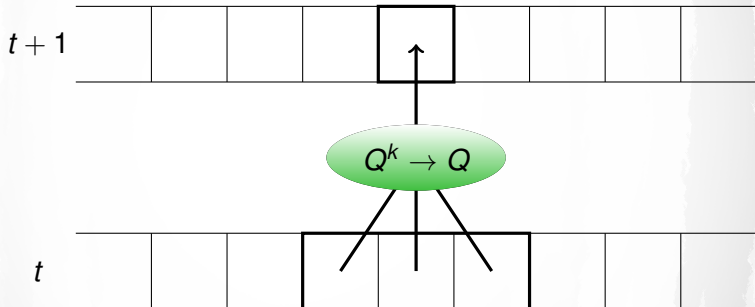
1 Definitions

2 Decision problems

3 Simulations

Deterministic CA

- states Q , k neighbors

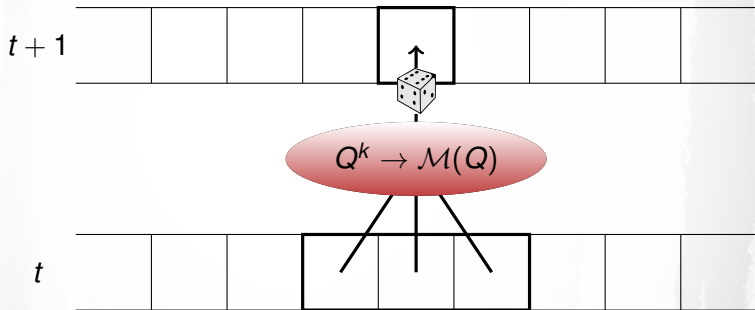


Hedlund's Theorem

Locally defined \Leftrightarrow continuous shift-commuting

Correlation-free probabilistic CA

- states Q , k neighbors



- global function: $Q^{\mathbb{Z}} \rightarrow \mathcal{M}(Q^{\mathbb{Z}})$ then $\mathcal{M}(Q^{\mathbb{Z}}) \rightarrow \mathcal{M}(Q^{\mathbb{Z}})$
- **examples:** noisy CA, α -asynchronous CA

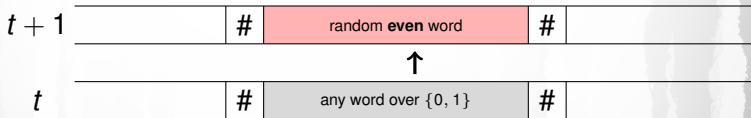
Limitations



Arrighi-Fargetton-Nesme-Thierry (CiE'11)

Applying Causality Principles to the Axiomatization of Probabilistic Cellular Automata

- 1 no stability by iteration
 - image by F of uniform conf. is a Bernoulli distribution
 - **not true** for F^2 in general
- 2 cannot generate random words of even parity

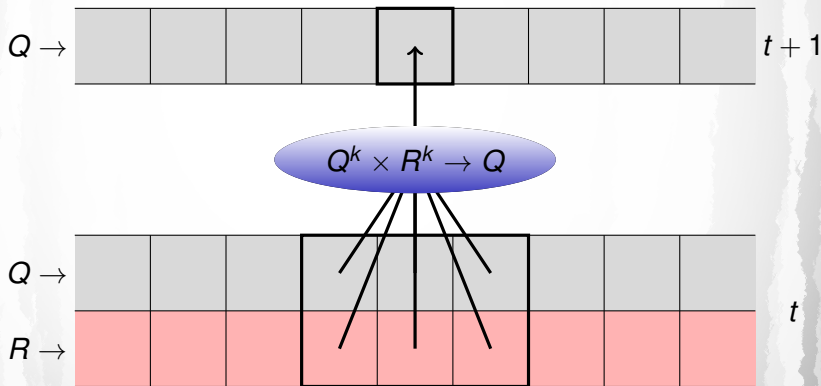


Fact

No correlation-free probabilistic CA can realize this function

Proposed formalism

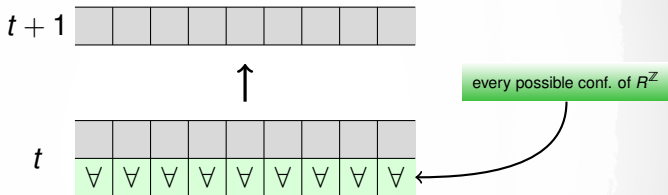
- (classical) states Q , k neighbors
- random states: R



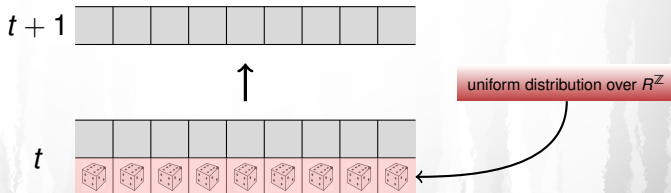
- explicit global map $F : Q^{\mathbb{Z}} \times R^{\mathbb{Z}} \rightarrow Q^{\mathbb{Z}}$

From local to global

1 non-deterministic global function $N_F : Q^Z \rightarrow \mathcal{P}(Q^Z)$



2 stochastic global function $S_F : Q^Z \rightarrow \mathcal{M}(Q^Z)$



Usefulness of local correlations

- global iterations

- by locality, we define (canonically) $S_F^t : Q^{\mathbb{Z}} \rightarrow \mathcal{M}(Q^{\mathbb{Z}})$

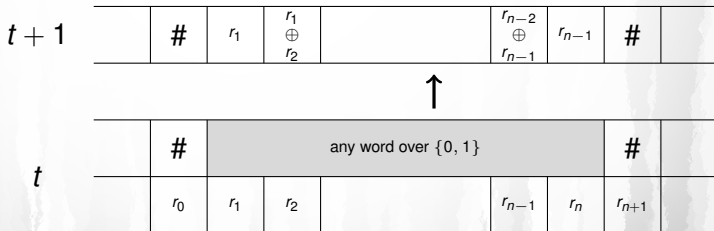
- **Fact:** if $S_f = S_G$ then $S_F^t = S_G^t$

- generating random even words

- $Q = \{\#, 0, 1\}$

- $R = \{0, 1\}$

- $k = 3$



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From local to global

- **syntactic object:** local map $f : Q^k \times R^k \rightarrow Q^k$
- **semantic object:** global map $S_F : Q^{\mathbb{Z}} \rightarrow \mathcal{M}(Q^{\mathbb{Z}})$

Given a syntactic object,
can we decide properties about
the corresponding semantic object

- **example:** equality of global maps
 - *input:* f and g
 - *question:* do we have $S_F = S_G$?

Equality of global maps

A non-trivial problem

Theorem

In **dimension 2 and higher**, equality of stochastic global maps is **undecidable**

- reduction from surjectivity problem for deterministic CA
- surjective \Leftrightarrow preserves uniform distribution
- also **undecidable** for **non-deterministic** global maps

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Fact

For **correlation-free** probabilistic CA, this problem is **decidable** in any dimension

Equality of global maps

Equality vs. coupling

■ point-wise coupling of global maps

- $F : Q^Z \times R_F^Z \rightarrow \mathcal{M}(Q^Z)$
- $G : Q^Z \times R_G^Z \rightarrow \mathcal{M}(Q^Z)$

Coupling

F and G are **coupled** on c by $\gamma \in \mathcal{M}(R_F^Z \times R_G^Z)$ if:

- 1 $\pi_1(\gamma)$ is the uniform distribution over R_F^Z
- 2 $\pi_2(\gamma)$ is the uniform distribution over R_G^Z
- 3 $\gamma(\{(r_F, r_G) \in R_F^Z \times R_G^Z : F(c, r_F) = G(c, r_G)\}) = 1$

Equality of global maps

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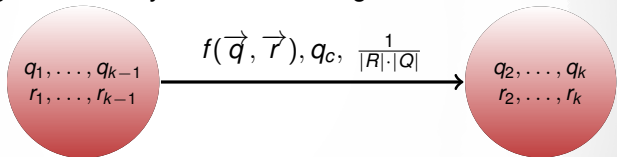
Theorem

$S_F = S_G$ if and only if F and G are coupled on each $c \in Q^{\mathbb{Z}}$.

Equality of global maps

1D, the realm of finite automata

- given $f : Q^k \times R^k \rightarrow Q$
- finite **probabilistic de Bruijn automaton** \mathcal{A}_F :
 - vertex set: $Q^{k-1} \times R^{k-1}$
 - edges labeled by $Q \times Q$, and weighted:

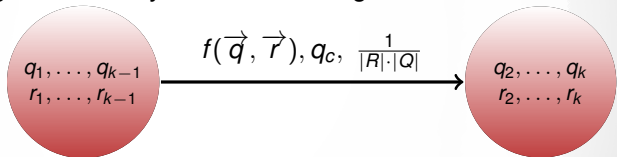


- *NB: initialization part omitted here for simplicity*

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Fact

$$L(\mathcal{A}_F) = L(\mathcal{A}_G) \Leftrightarrow S_f = S_G$$

Corollary

Equality of stochastic global maps is **decidable** in 1D.

Büchironnerie

Model checking for 1D Non-deterministic CA

- known proof technique for 1D deterministic CA



K. Sutner, 2009

Model Checking One-Dimensional Cellular Automata



O. Finkel, 2011

On Decidability Properties of One-Dimensional Cellular Automata

Key result

ω -automatic structures have a decidable first-order theory.

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ω -automatic structures have a decidable first-order theory.

- extension to non-deterministic 1D CA
 - reachability predicate: $x \rightarrow y \equiv y \in N_F(x)$
 - $(\mathbb{Q}^{\mathbb{Z}}, \rightarrow)$ has a **decidable** first-order theory
- *injectivity*: $\forall x, y, z : (x \rightarrow z \wedge y \rightarrow z) \Rightarrow x = y$
- *determinism*: $\forall x, y, z : (x \rightarrow z \wedge x \rightarrow y) \Rightarrow y = z$
- *being noisy*: $\forall x, y : x \rightarrow y$

1D, still some undecidability

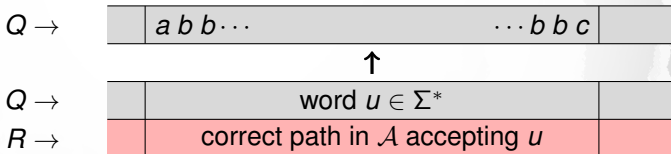
Simulating probabilistic automata by stochastic CA

- **given** a finite probabilistic automaton \mathcal{A}
 - alphabet Σ , states $Q_{\mathcal{A}}$
 - $N = \text{lcm}$ of denominators of (rational) transition probas
- **construct** F with
 - $Q = \Sigma \cup \{0, a, b, c\}$
 - $|R| = N \cdot |Q_{\mathcal{A}}|$
 - random $r \in R \leftrightarrow$ random pair (vertex, outgoing edge)

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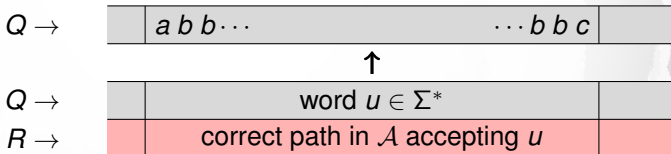


- output 0 in any other case

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- output 0 in any other case

Fact

For any $u \in \Sigma$, we have $\Pr\{F(u) = ab^{|u|}c\} = \Pr_{\mathcal{A}}(u) \cdot |Q_{\mathcal{A}}|^{-|u|}$

1D, still some undecidability

Emptiness problem

- given \mathcal{A} and $\lambda > 0$, is there u such that $Pr_{\mathcal{A}}(u) > \lambda$?



Blondel-Canterini, 2003

Undecidable Problems for Probabilistic Automata of Fixed Dimension

- undecidable problem, even for automata over $\{0, 1\}$ with 46 states

1D, still some undecidability

Emptiness problem

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Blondel-Canterini, 2003

Undecidable Problems for Probabilistic Automata of Fixed Dimension

- undecidable problem, even for automata over $\{0, 1\}$ with 46 states

Corollary

There is some α such that the following problem is undecidable in 1D: given a stochastic CA F and $\lambda > 0$, is there some word u such that

$$Pr\{F(u) = ab^{|u|}c\} > \lambda \cdot \alpha^{|u|}$$

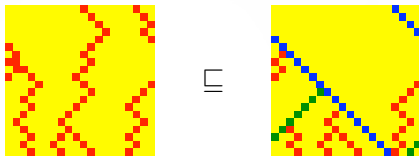
- the non-probabilistic version is decidable
- conjectured also decidable for correlation-free CA

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Restrictions/projections

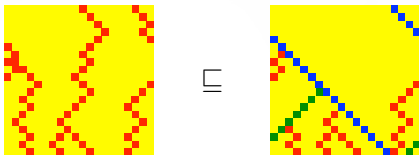
- **restriction** to a stable subset of states: $F \sqsubseteq G$



$i : Q_F \rightarrow Q_G$ with $i(G'(c, r)) = G(i(c), r)$ and $S_{G'} = S_F$

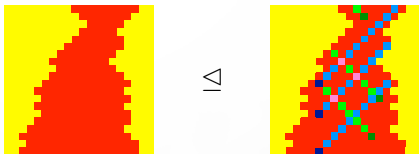
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- compatible **projection**: $F \trianglelefteq G$

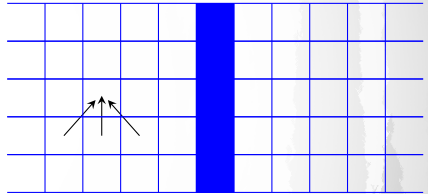


$\pi : Q_G \rightarrow Q_F$ with $G'(\pi(c), r) = \pi(G(c, r))$ and $S_{G'} = S_F$

- any combination of both: projection after restriction $\trianglelefteq \sqsubseteq$

Rescaling

► 3 parameters: $F \mapsto F^{(m,t,z)}$



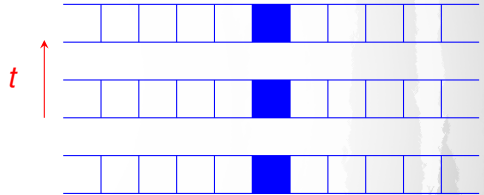
Global map

$$F^{(1,1,0)} = F$$

Rescaling

▶ 3 parameters: $F \mapsto F^{(m,t,z)}$

■ time



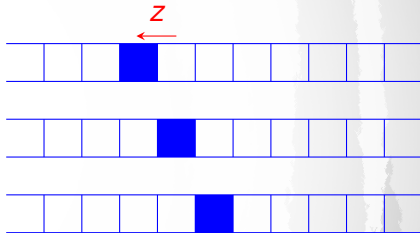
Global map

$$F^{(1,t,0)} = F^t$$

Rescaling

► 3 parameters: $F \mapsto F^{(m,t,z)}$

- time
- shift



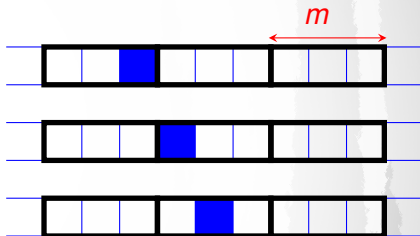
Global map

$$F^{(1,t,z)} = \sigma_z \circ F^t$$

Rescaling

► 3 parameters: $F \mapsto F^{(m,t,z)}$

- time
- shift
- cell grouping



Global map

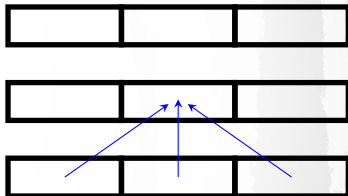
$$F^{(m,t,z)} = \mathbf{o}_m^{-1} \circ \sigma_z \circ F^t \circ \mathbf{o}_m$$

$$\left. \begin{array}{l} \mathbf{o}_m : (Q^m)^{\mathbb{Z}} \rightarrow Q^{\mathbb{Z}} \\ \mathbf{o}_m : (Q^m)^{\mathbb{Z}} \times (R^m)^{\mathbb{Z}} \rightarrow Q^{\mathbb{Z}} \times R^{\mathbb{Z}} \end{array} \right\} \text{canonical bijections}$$

Rescaling

► 3 parameters: $F \mapsto F^{(m,t,z)}$

- time
- shift
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Global map

$F^{(m,t,z)}$ is a cellular automaton

- with possibly different state sets (Q and R)
- with a possibly different neighborhood

Simulation and universality

- simulation \equiv local comparison **up to** rescaling
- $F \preceq_i G$ iff $\exists m, m', t, t', z, z' : F^{(m,t,z)} \sqsubseteq G^{(m',t',z')}$
- $F \preceq_\pi G$ iff $\exists m, m', t, t', z, z' : F^{(m,t,z)} \trianglelefteq G^{(m',t',z')}$
- analogously, define \preceq_i^N and \preceq_π^N for non-deterministic maps
(asking $N_{G'} = N_G$ instead of $S_{G'} = S_G$ in def. of \sqsubseteq and \trianglelefteq)
- **Fact:** if $F \preceq G$ then $F \preceq^N G$ for $\preceq = \preceq_i$ or \preceq_π

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Universality

Let \preceq be any of \preceq_i , \preceq_π , \preceq_i^N or \preceq_π^N .

G is \preceq -**universal** if for all F it holds $F \preceq G$

Universality

- **definition:** $\mathcal{PF}(F)$ is the set of prime factors of $|R_F|$

Proposition

If F and G are not deterministic and $F \preceq_i G$ (or $F \preceq_\pi G$) then $\mathcal{PF}(F) \cap \mathcal{PF}(G) \neq \emptyset$

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Corollary

There is no \preceq_i -universal CA (same for \preceq_π).

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Theorem

- 1 There is a \preceq_i^N -universal CA.
- 2 for any set of prime number P , there is a **correlation-free** CA G such that for any F

$$\mathcal{PF}(F) \subseteq P \Rightarrow F \preceq_i G$$

Correlation-free vs. general CA

Proposition

For any F there is a **correlation-free** G such that $F \sqsubseteq G^2$

- **undecidable in 2D:** F, G correlation-free, do we have $S_F^2 = S_G^2$?
- can we do anything with 2 steps of a correlation-free CA?

Correlation-free vs. general CA

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- can we do anything with 2 steps of a correlation-free CA?

NO!

There is no correlation-free F and t such that F^t is the 'random even word function'.

- the set of space-time diagrams ($\subseteq Q^{\mathbb{N} \times \mathbb{Z}}$) of a correlation-free CA is a SFT
- the set of space-time diagrams of a CA is a sofic shift

Questions

Questions

- is there for any F a correlation-free G such that

$$F \trianglelefteq G$$

- \preceq_i equivalence classes without correlation-free CA?
- more on decidability/undecidability in stochastic 1D maps
- continuity/computability of coupling
- find a formalism of PCA with a Hedlund theorem

Merci !