Stabilizing SFT by Cellular Automata

Séminaire CANA

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Credits

- **[FMT21]:** "*Self-stabilisation of cellular automata on tilings*" by **N. Fatès, I. Marcovici** and **S. Taati**
- **unpublished joint work** with **M. Delacourt, P. Guillon, N. Ollinger**
- earlier **discussions** with **A. Gajardo, D. Maldonado** and **S. Taati**

Subshift of finite type (SFT)

- A : finite **alphabet**
- $A^{\mathbb{Z}^d}$: d-dimensional **configurations**
- **patterns** : $p : D \to A$ with $D \subseteq \mathbb{Z}^d$ **finite**
- **pattern occurrence** : $p \sqsubset c$ if $\exists z$ with $c(z+z') = p(z')$ for all $z' \in D$

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- in this talk (wlog), SFT using only domino domains:
  - 1D: $D = \{0,1\}$
  - 2D: $D = \{(0,0),(1,0)\}$ and $D = \{(0,0),(0,1)\}$

Cellular automata (CA)

- defined by a **local map** $\lambda : A^D \to A$
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- **asymptotic** configurations: $x \overset{\infty}{=} y$ if $\{z : x(z) \neq y(z)\}$ is finite

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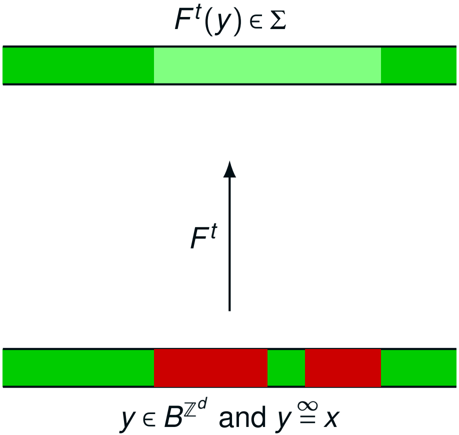

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Theorem (Gács-Kurdyumov-Levin-78)

The following 1D CA stabilizes Σ:

$$F(x)_z = \begin{cases} maj(x_z, x_{z+1}, x_{z+3}) & \text{if } x_z = 1 \\ maj(x_z, x_{z-1}, x_{z-3}) & \text{else.} \end{cases}$$

Local errors VS. distance to Σ

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- number of **local errors**

$$\epsilon(y) = \#\{z : y(z)y(z+\vec{e}) \in H \text{ or } y(z)y(z+\vec{n}) \in V\}$$

- **distance** to Σ

$$d(y,\Sigma) = \min\{|D| : \exists x' \in \Sigma, \forall z \notin D, y(z) = x'(z)\}$$

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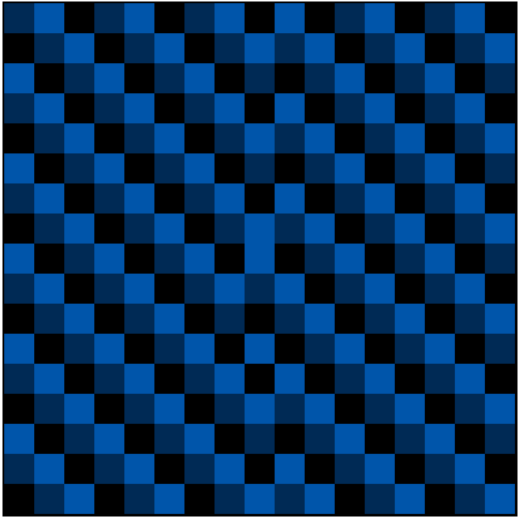
- ⚠ we can have $\epsilon(y) \ll d(y,\Sigma)$
- a stabilizer must be able to **change arbitrary large patches of locally valid cells**!
- **stabilization time** as a function of $d(y,\Sigma)$

Example: Σ = 3-colorings



Previous results from [FMT21]

Theorem [FMT21]

Any non-wandering 1D SFT is stabilizable in linear time.

- *intuition:* can be patched from left to right.

Theorem [FMT21]

Any NE-deterministic 2D SFT is stabilizable.

- *intuition:* can be reduced to a 1D patching process using the determinism.

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Theorem 1

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- $B = A \cup A \times M$: cells in $A \times M$ are said **marked**

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- **zone** = maximal 4-connected region of marked cells

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- $B = A \cup A \times M$: cells in $A \times M$ are said **marked**
- **zone** = maximal 4-connected region of marked cells
- what the CA F do:
  1. **mark** any locally invalid cell
  2. **normalize** zones (to be defined in next slide)
  3. brute force **computation** inside normalized zone
     - if valid patch found → remove the marks
     - if not → extend zone by 1 cell in each direction

Normalized zones

L	←	←	←	←	←
→	→	→	→	→	↑
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- Turing computation inside: *try all possible A-filling with backtrack*
- no computation loop, even with bad initialization

Demo

- Σ = 3-colorings in 2D
- a CA F with 69 states and radius 1 that stabilizes it

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4. impossible to have a constant number of zones ≥ 2 forever
5. when a single zone remains, it unmarks itself once large enough

δ-stabilizing under ϵ-perturbations

- $A = \{0, \ldots, k-1\}$
- μ Bernoulli measure over $A^{\mathbb{Z}^d}$ with $\mu(0) \geq 1-\epsilon$
- ρ a μ-**random configuration**
- ϵ-**perturbation** of x: $x + \rho = z \mapsto x(z) + \rho(z) \mod k$

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F δ-stabilizes Σ under ϵ-perturbation if:

- $\forall x \in \Sigma, F(x) = x$
- $\forall x \in \Sigma$, for μ-almost all ρ:
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density $\leq \epsilon$

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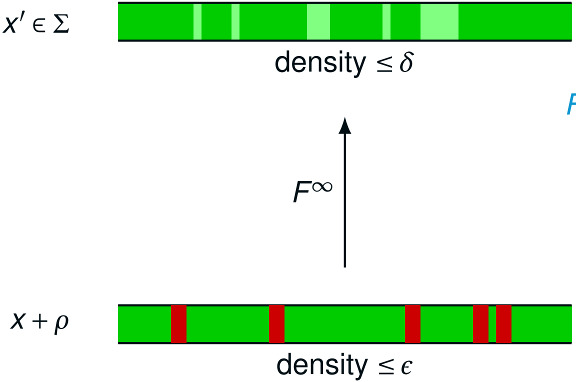

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If Σ can be stabilized in **linear time**, then
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Theorem [FMT21]

Unless P=NP, there is Σ in 2D which cannot be stabilized in polynomial time.

New: general linear time stabilization in 1D

Theorem 2

Any SFT $\Sigma \subseteq A^{\mathbb{Z}}$ is stabilizable in **linear time.**

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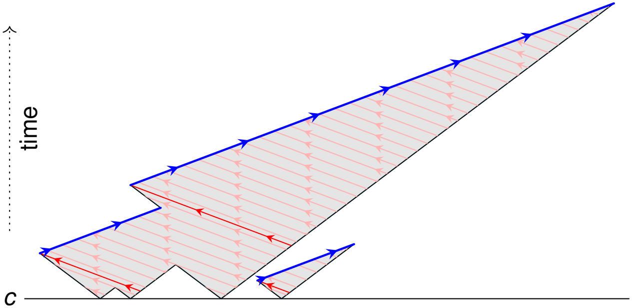
Corollary

For any 1D SFT Σ, $\forall \delta, \exists \epsilon$ such that Σ can be δ-stabilized against ϵ-perturbations.

Construction idea

- $B = A \sqcup M$ with $M \sim 2^A$
- radius 2



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2. ←: update set of possible states compatible with what's on right
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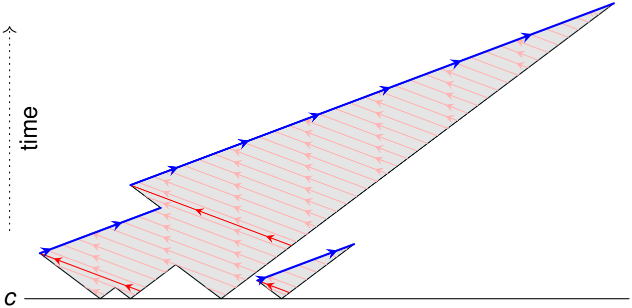

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- ⚠ exact definition & analysis a bit subtle...

Work in progress & future work

- if context-filling problem of Σ is **polytime** then stabilization in **polytime**
- remove extra symbols (*i.e.* $B = A$)
- which SFT have a polytime context-filling problem?
- what about random perturbation in 2D? e.g. with 3-colouring?