Domino and cellular automata problems on graphs

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- MSO decidable on $Cayley(\Gamma) \iff \Gamma$ virtually free [Kuske-Lohrey, 2005]
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Introduction

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- proved true on several classes of groups:

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- **this talk:** more problems and more graphs

- $G = (V, E, \Delta)$ digraph, edges labeled by finite set Δ
- $E = \bigcup_{\delta \in \Delta} E_\delta$
- Q : finite set of states/symbols/colors
- Q^V : configurations

Graphs and configurations

- $G = (V, E, \Delta)$ digraph, edges labeled by finite set Δ
- $E = \bigcup_{\delta \in \Delta} E_\delta$
- Q : finite set of states/symbols/colors
- Q^V : configurations
- **domino constraints** $\mathcal{D} = (D_\delta)_{\delta \in \Delta}$ with $D_\delta \subseteq Q^2$
- set of $\mathcal{D}$-**valid configurations**:

$$\Sigma_{\mathcal{D}} \stackrel{\text{def}}{=} \{c : (v, v') \in E_\delta \implies (c(v), c(v')) \in D_\delta\}$$

- **example:** k-coloring $\iff$ $D_\delta = \{(q, q') : q \neq q'\}$ and $Q = \{1, \ldots, k\}$

`domino` problem

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1-domino problem

- input: $\mathcal{D} = (D_\delta)_{\delta \in \Delta}$ and $s \in Q$
- question: $\exists c \in \Sigma_{\mathcal{D}}^{s,1} = \left\{c \in \Sigma_{\mathcal{D}} : \#\{v : c(v) = s\} \geq 1\right\}$?

∞-domino problem

- input: $\mathcal{D} = (D_\delta)_{\delta \in \Delta}$ and $s \in Q$
- question: $\exists c \in \Sigma_{\mathcal{D}}^{s,\infty} = \left\{c \in \Sigma_{\mathcal{D}} : \#\{v : c(v) = s\} = \infty\right\}$?

Deterministic graphs

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- *notation:* $v \odot \delta = \begin{cases} v' & \text{if } (v, v') \in E_\delta \\ \bot & \text{else.} \end{cases}$
- $v \odot u \in V \cup \{\bot\}$ for $v \in V$ and $u \in \Delta^*$

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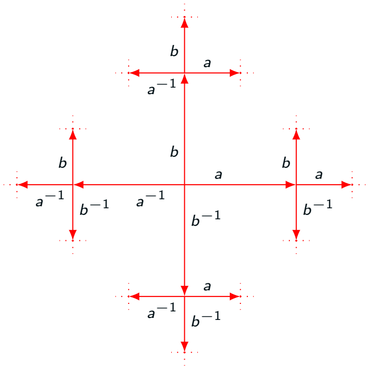

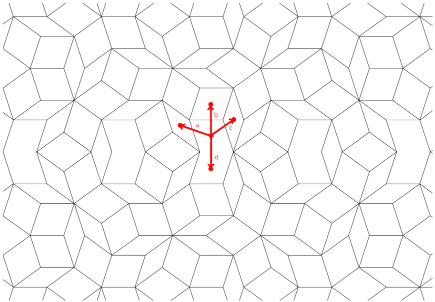

- $\Delta^{\leq k} = \{u \in \Delta^* : |u| \leq k\}$
- **local patterns** of radius k: $\mathcal{P}_k \stackrel{def}{=} \Delta^{\leq k} \to Q \cup \{\bot\}$
- **local pattern at** v **in** c: $p_k(c, v) = u \in \Delta^{\leq k} \mapsto c(v \odot u)$

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- let $\mathcal{F} \subseteq \mathcal{P}_k$, the associated CSFT (closed set of finite type) is

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Higher block recoding

`domino` $\equiv$ CSFT emptiness problem

- **local rule** of radius k: $\lambda : \mathcal{P}_k \to Q$
- associated **cellular automaton** $F_\lambda : Q^V \to Q^V$

$$\forall c \in Q^V, \forall v \in V : F_\lambda(c)_v = \lambda\big(p_k(c, v)\big)$$

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- **FO logic on orbits:** $FO(=, \to)$
  - *fixed point*: $\exists x, x \to x$
  - *injectivty*: $\forall x, y, z : (x \to z \wedge y \to z) \implies x = y$
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- **problem** $\texttt{check}(\phi)$
  - input: λ
  - question: $F_\lambda \models \phi$?

Remark

$\texttt{check}(\phi)$ independent of choice of generators for Cayley graphs

Equivalences

- `domino` $\equiv \phi_d$-check *(fixed-point)*
- `1-domino` $\equiv \phi_1$-check *(fixed-point with pre-image)*
- ∞`-domino` $\equiv \phi_\infty$-check *(fixed-point with non-asymptotic pre-image)*

- $\phi_d \stackrel{def}{=} \exists x, x \to x$
- $\phi_1 \stackrel{def}{=} \exists x, y, x \to x \wedge y \to x \wedge x \neq y$
- $\phi_\infty \stackrel{def}{=} \exists x, y, x \to x \wedge y \to x \wedge \neg(x \stackrel{\infty}{=} y)$
- where $x \stackrel{\infty}{=} y \iff \{v : x(v) \neq y(v)\}$ finite

- graph minor of G: deleting vertices or edges and contracting edges

A hard FO formula

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Theorem (T., 2024)

There is a $FO(=, \to)$ formula ϕ_0 such that the problem $\texttt{check}(\phi_0)$ is Σ^1_1-**hard** on any bounded degree graph having the $\mathbb{N} \times \mathbb{N}$-grid as minor.

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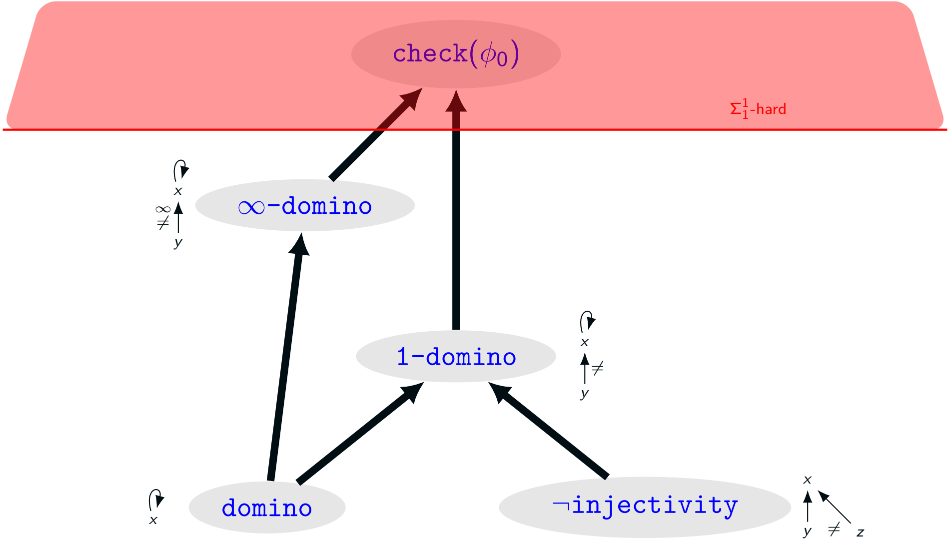
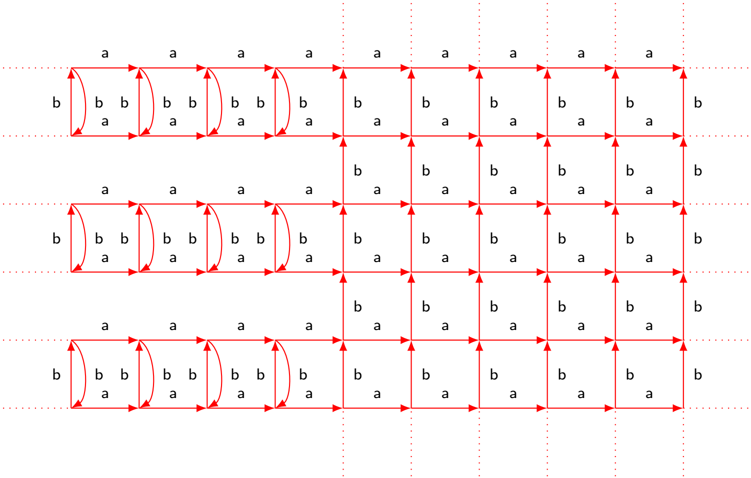
Corollary

$\texttt{check}(\phi_0)$ is decidable on a group Γ **iff** Γ is virtually free.

- *reminder:* for $G = \mathit{Cayley}(\Gamma)$, having $\mathbb{N} \times \mathbb{N}$-grid minor $\iff$ Γ not virtually free.

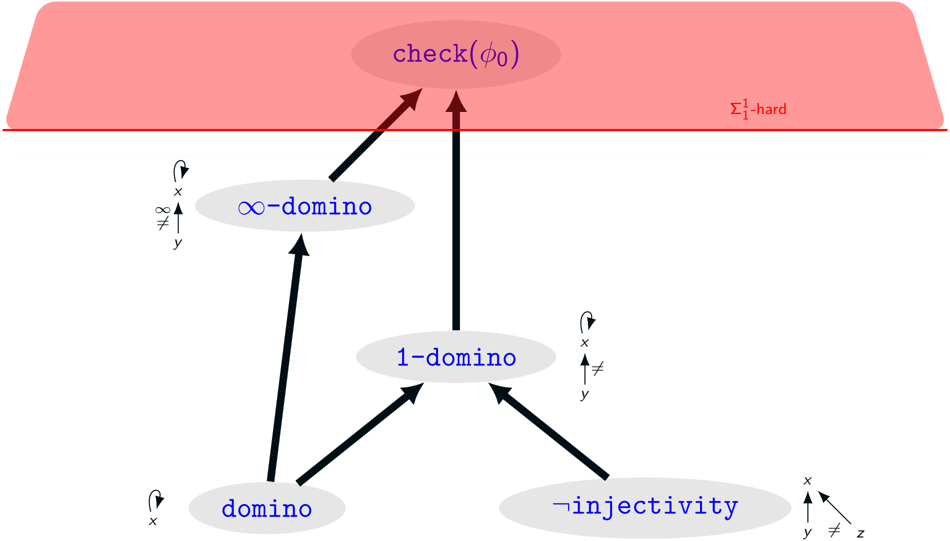
G deterministic with $\mathbb{N} \times \mathbb{N}$-grid minor

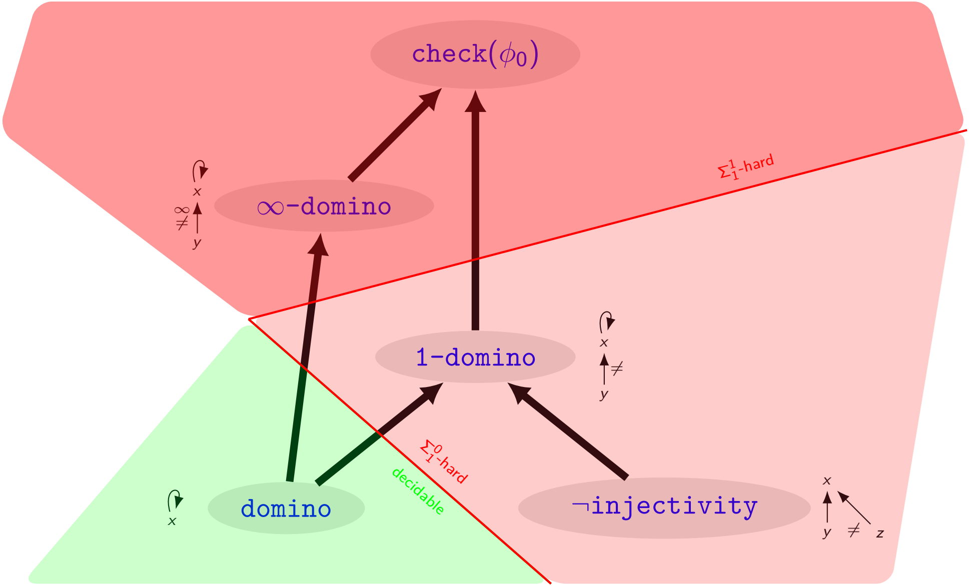




$$\Delta = \{a, b, a^{-1}, b^{-1}\}$$

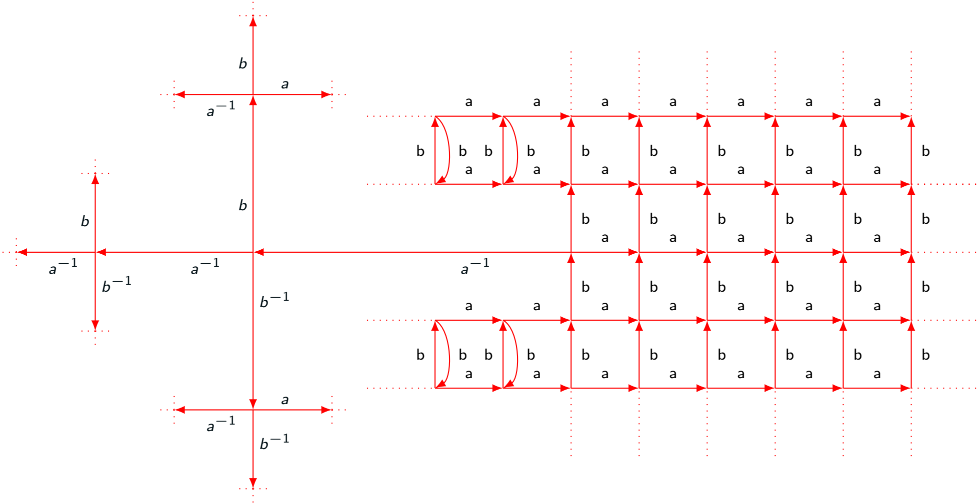
Pytheas graph G_1





check(ϕ_0)
∞-domino
1-domino
domino
¬injectivity
Σ^1_1-hard
Σ^0_1-hard
decidable
x
y
∞
≠
z

Pytheas graph G_2



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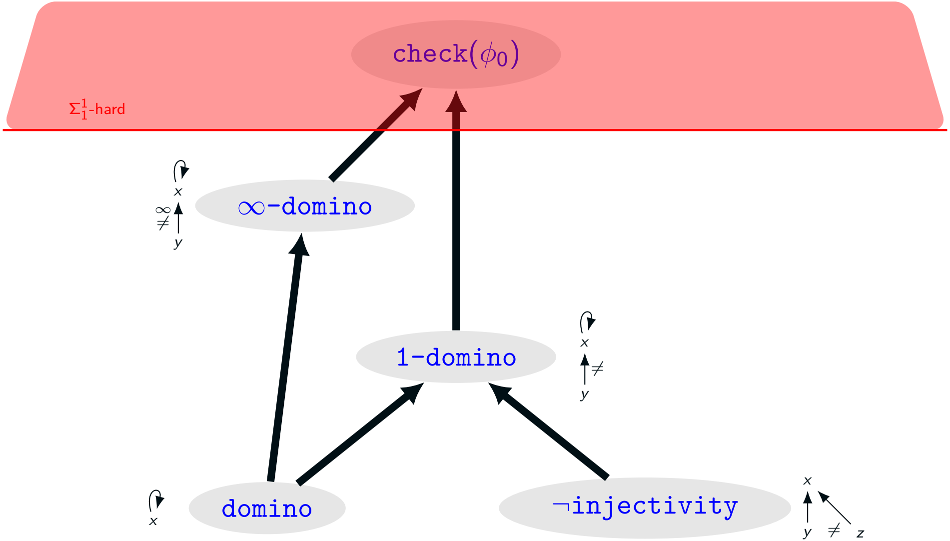
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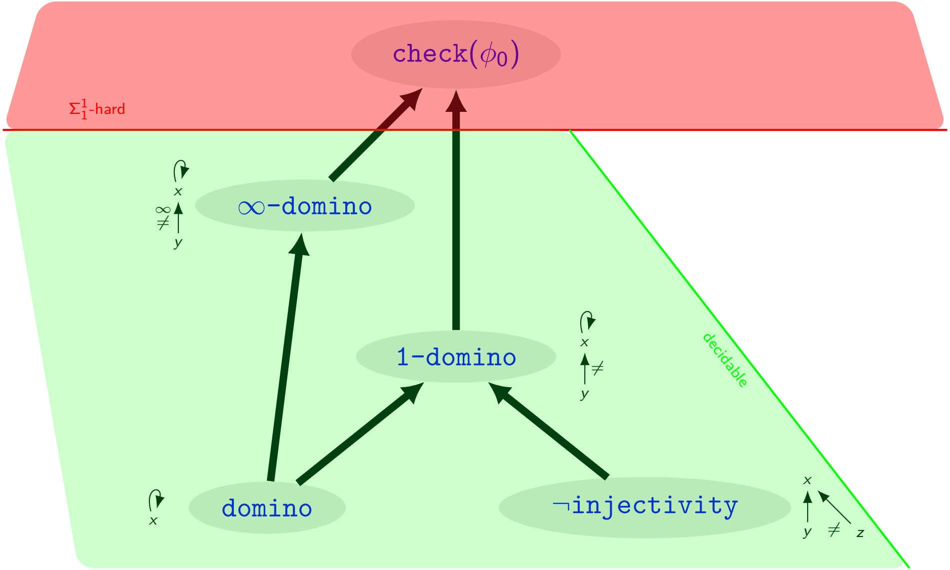
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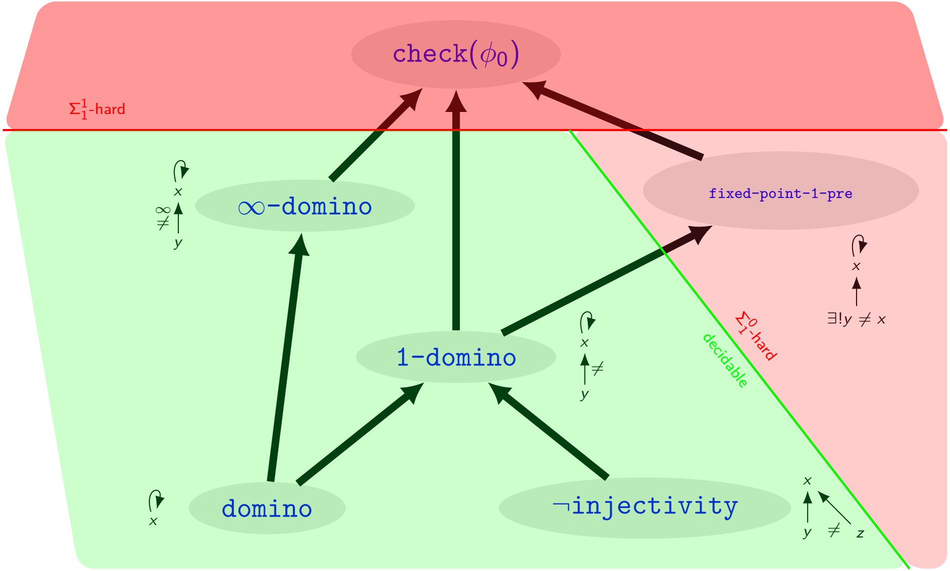

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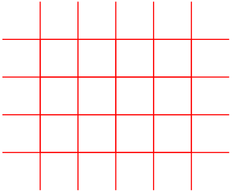

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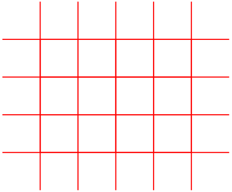

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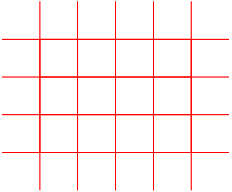

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fixed-point undecidable

Theorem

check(ϕ_0) still Σ^1_1-**hard** on bounded degree graphs with $\mathbb{N} \times \mathbb{N}$-grid minor

- [Cohen,2017]
- [Arrighi-Durbec-Guillon, 2023]
- [Hellouin-Lutfalla-Vanier,2025]
- generalization of / link with the above: **Nicolas's talk tomorrow!**

Exercise

Find G_1, G_2 quasi-isometric with `fixed-point` decidable on G_1 but not on G_2.

Questions

1. for Cayley graphs of groups: decidability of surjectivity?

2. Pytheas style counter-examples for Cayley graphs of semi-groups? monoids?

3. `fixed-point` undecidable for any Uyuni graph?

   i.e. infinite planar bounded degree graph quasi-isometric to $\mathbb{Z}^2$?