Monotone 1D CA acting on probability measures

31st AUTOMATA workshop

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Your undecidability results have no physical meaning!

Me, 20 years ago

Physicist

1D CA on probability measures

- Q finite **alphabet** / $Q^{\mathbb{Z}}$ **configurations**
- **CA global rule** $F : Q^{\mathbb{Z}} \to Q^{\mathbb{Z}}$ / **shift map** $\sigma_k(x)_z = x_{z-k}$

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- μ completely determined by its value on cylinders: $[u] = \{x \in Q^{\mathbb{Z}} : x_1 \cdots x_{|u|} = u\}$

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- **action of CA** F on measure μ: $F\mu(E) = \mu(F^{-1}(E))$
- **orbit**: $\mu \to F\mu \to F^2\mu \to \cdots F^t\mu \to \cdots$

Undecidability for 1D CA on probability measures

- F is μ-**nilpotent** if $F^t\mu([0]) \to_t 1$

Theorem (Boyer-Delacourt-Poupet-Sablik-T.-2015)

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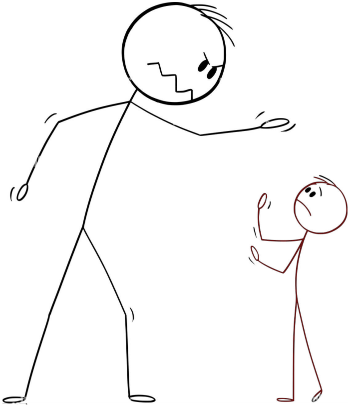
- μ **computable** if there is $\texttt{algo}(u,n)$ computing $\mu([u]) \pm \frac{1}{n}$
- μ **limit computable** if limit of computable measures: $\mu_n \to \mu$

  $\left(i.e.\ \mu_n([u]) \to \mu([u]) \text{ for all } u\right)$

Theorem (Hellouin-Sablik-2016)

$F^t\mu_0 \to_t \mu$ for some $F \iff \mu$ limit-computable

We got undecidability results for random initial configurations!



Me, 10 years ago

Physicist

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Me, last year

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Main question

Action of **monotone** CA on **probability measures** as complex as in the general case?

- μ **translation-ergodic** if $\mu(E) \in \{0,1\}$ whenever $\sigma_z(E) = E$
- $\equiv$ "u appears with frequency $\mu(u)$ in any μ-random configuration"
- Choquet's theorem: *any translaiton-invariant measure is a "convex combination" of translation-ergodic measures*
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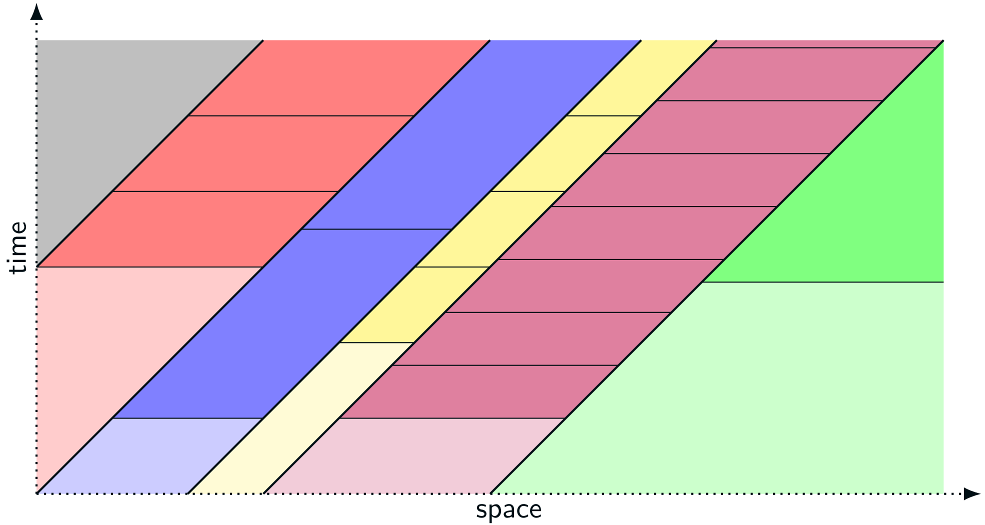
Theorem 1

For any computable translation-ergodic full-support μ and any boolean monotone 1D CA

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t\leq n} F^t \mu$$

exists and is computable.


time
space

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There is a boolean monotone 1D CA for which the Cesaro limit μ^∞ starting from the uniform measure has uncomputable support.

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Theorem 3

There is a boolean monotone 1D CA such that $(F^t\mu)_t$ is not simply convergent for μ the uniform measure.

Theorem 3

“conveyor belt trick” + “antichain trick” + “anaphase”