On Kurka's Dichotomy for Cellular Automata on Groups

31st AUTOMATA workshop

J. A. H. Audouard, G. Theyssier

Institut de mathématiques de Marseille
(CNRS, Université Aix-Marseille)

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INSTITUT de MATHÉMATIQUES de MARSEILLE

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Sensitivity to initial conditions

$\exists \epsilon, \forall x, \forall \delta, \exists y \in B_\delta(x), \exists t : F^t(y) \notin B_\epsilon(F^t(x))$

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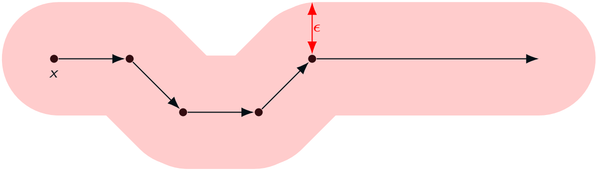

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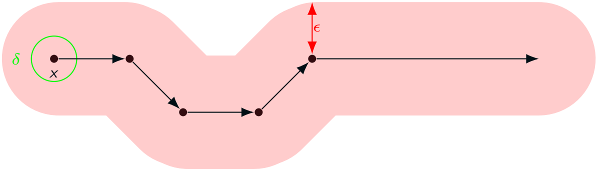

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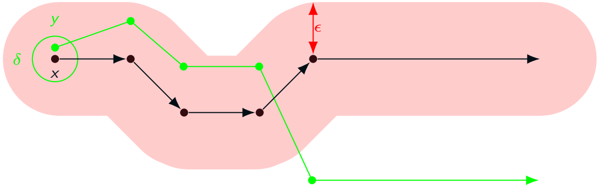

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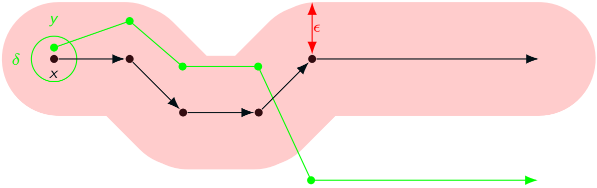

$\exists \epsilon, \forall x$: $orbit(x)$ **is not** ϵ-stable

Equicontinuity point

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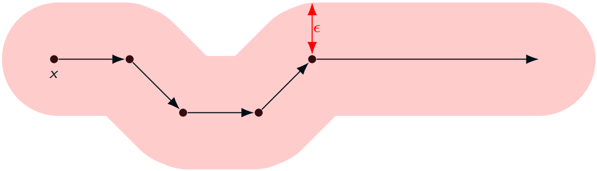

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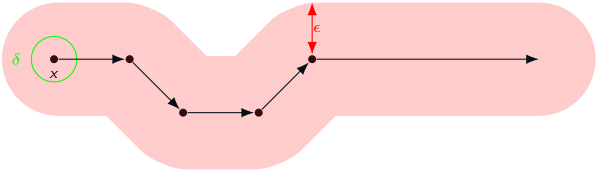

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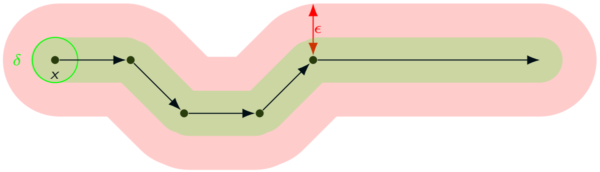

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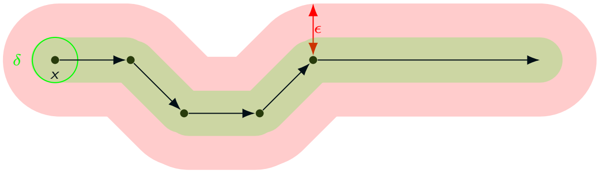

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A dichotomy?

Sensitivity

$\exists \epsilon, \forall x$: *orbit*(x) **is not** ϵ-stable

vs

Equicontinuity point

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A dichotomy?

Sensitivity vs **Equicontinuity point**

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- equicontinuity point $\implies$ not sensitive
- converse do not hold for dynamical systems in general

Sensitivity vs **Equicontinuity point**

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Theorem (Kurka's dichotomy, 1997)

Any 1D CA is either sensitive or has (a lot of) equicontinuity points.

Sensitivity vs **Equicontinuity point**

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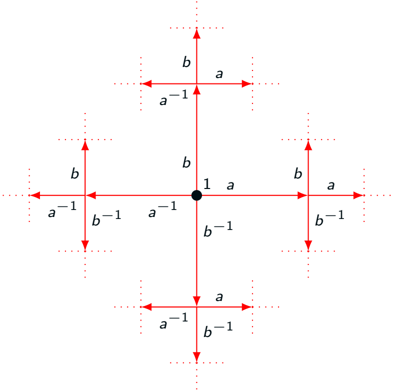
Theorem (Kurka's dichotomy, 1997)

Any 1D CA is either sensitive or has (a lot of) equicontinuity points.

Theorem (Sablik-T., 2008)

There exists a 2D CA which is not sensitive and has no equicontinuity point.



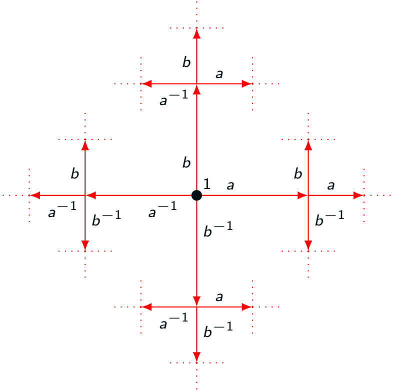
$E = \{a, b, a^{-1}, b^{-1}\}$

- $G =< E >$ a group
- E: finite set of generators
- Cayley graph: vertices G and edges

$$\{(g, g \cdot e) : g \in G, e \in E\}$$

CA on Groups

Configurations and metrics



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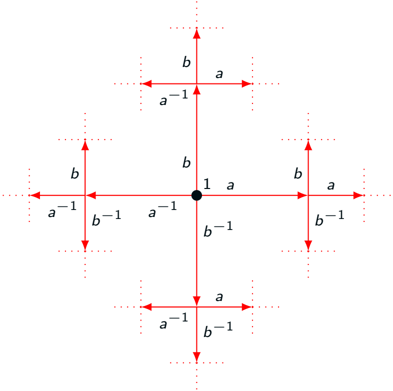

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- Q finite alphabet / **configurations:** Q^G
- d^E : **word metric** $\equiv$ distance in Cayley graph
- d_E: **Cantor metric** over Q^G

$$\delta(x, y) = \min_{g \in G}\{d^E(1, g) : x(g) \neq y(g)\}$$

$$d_E(x, y) = 2^{-\delta(x,y)}$$

- **neighborhood:** $V \subseteq G$ finite
- **local map:** $\lambda : Q^V \to Q$
- **global map:** $F : Q^G \to Q^G$ s.t.

$$\forall x \in Q^G, \forall g \in G, F(x)_g = \lambda\big((g^{-1} \cdot x)_{|V}\big)$$

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- *examples:*

- **XOR CA**
- $V = E$
- $\lambda(u) = \sum_{e \in E} u_e \bmod 2$

- **Identity CA**
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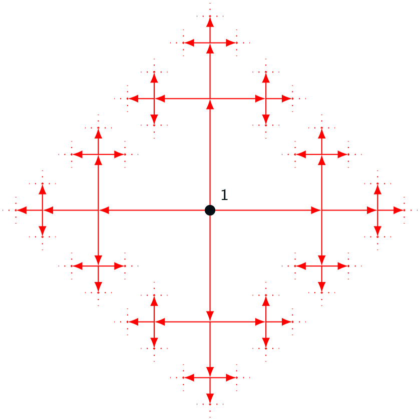
- *examples:*

- **XOR CA**
- $V = E$
- $\lambda(u) = \sum_{e \in E} u_e \bmod 2$
- Sensitive on $\mathbb{Z}$, $\mathbb{Z}^2$ and $\mathbb{F}_2$

- **Identity CA**
- $V = \{1\}$
- $\lambda(u) = u_1$
- All x are equicontinuity points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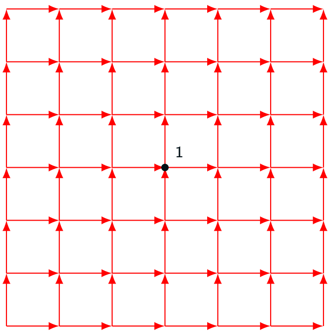
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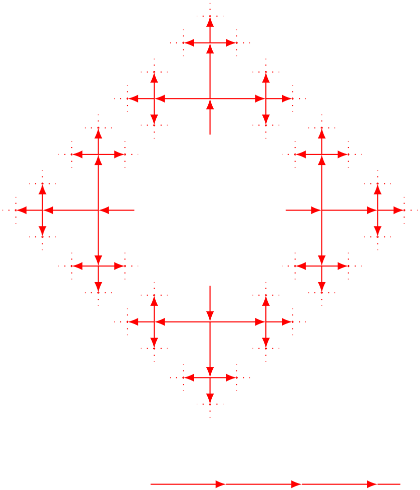

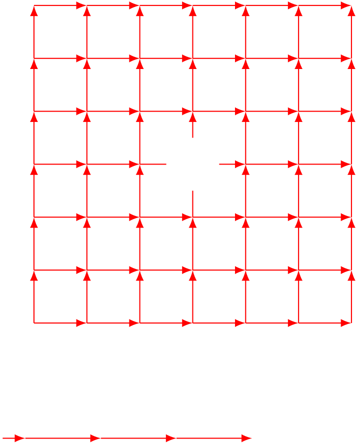

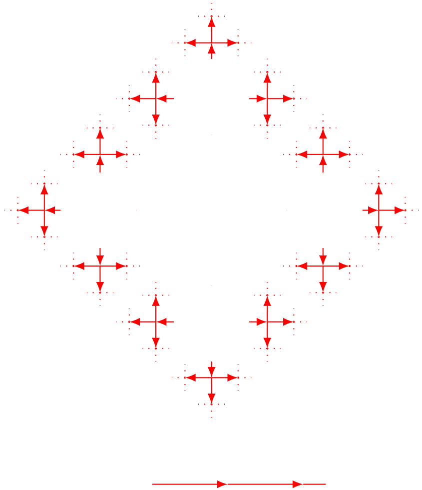

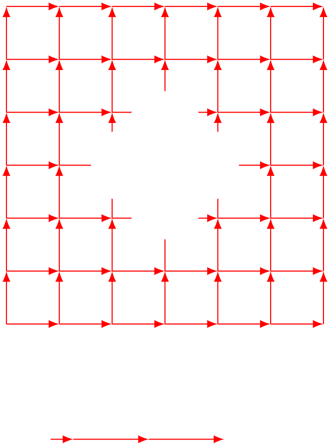

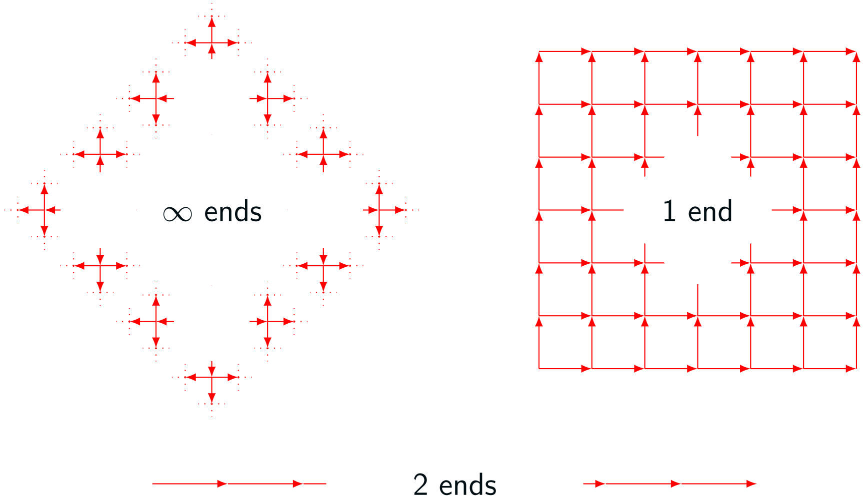

∞ ends
1 end
2 ends

On which group dichotomy holds?

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If dichotomy holds on G it must hold on any subgroup of G.

Theorem 1 (extending Kurka's argument)

Dichotomy holds on any group with 2 ends.

On which group dichotomy holds?

- 2 ends: **always**
- 1 end: not always (Theorem Sablik-T, 2008)
- ∞ ends?

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Dichotomy **does not hold** on a free group with ≥ 2 generators.

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"Bottle at sea" conjecture

Kurka's dichotomy holds exactly on groups with 2 ends

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- Gilman dichotomy: μ-sensitivity vs. μ-equicontinuity point

Theorem (Barbieri-García-Ramos-Taati, preprint 2024)

The Gilman dichotomy holds exactly on groups with 2 ends

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