FO logic on CA orbits equals MSO logic

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- $\mathcal{LV}$: finite set of possible **local views** (*def. on next slide*)
- $L_{v,c} \in \mathcal{LV}$: local view at v in configuration c
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CA global map $F : Q^V \to Q^V$

$$F(c)_v = f(L_{v,c})$$

Cellular Automata

- Δ = labels on edges
- k : lookup distance (locality)
- $\mathcal{LV} = \Delta^{\leq k} \to 2^Q$: set of possible **local views**

$$L_{v,c} = w \mapsto \{c_{v'} : v \xrightarrow{w} v'\}$$

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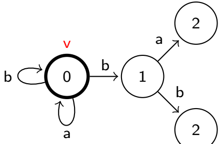
- $Q = \{0, 1, 2\}$
- $\Delta = \{a, b\}$
- $k = 2$



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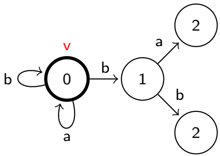

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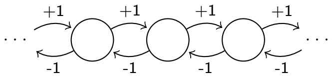
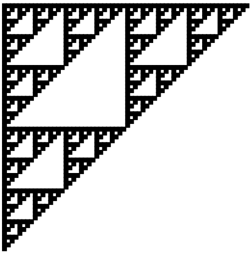
G **+ local rule** $f : \mathcal{LV} \to Q \rightsquigarrow$ **global map** $F_{G,f} : Q^V \to Q^V$

Cellular Automata

- $G = Cayley(\mathbb{Z}, \Delta)$ with $\Delta = \{-1, +1\}$



  - $Q = \{0, 1\}$
  - $k = 1$
  - $\mathcal{LV} = (L(-1), L(\epsilon), L(1)) \sim Q^3$
  - $L_{v,c} \sim (c_{v-1}, c_v, c_{v+1})$
  - $f(a, b, c) = a + b \bmod 2$



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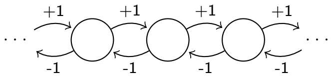

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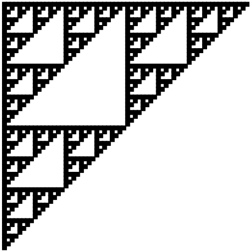

▷ **our def. is equivalent to the classical one on Cayley graphs**

FO logics on CA orbits

- $FO(=, \to)$
- variables $\equiv$ configurations
- $x \to y$ means "$F(x) = y$"
- **model checking** of ϕ on G (f given as input):

$$F_{G,f} \models \phi \ ?$$

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Fixed-point: $\exists x, x \to x$

- equivalent to the **domino problem** on Cayley graphs
- decidable on $\mathbb{Z}$ [Folklore]
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Conjecture [Ballier-Stein, 2013]

On Cayley graphs of f.g. groups:
domino problem decidable **IFF** MSO model checking decidable

FO logics on CA orbits

Injectivity/surjectivity

- *injectivity*: $\forall x, \forall y, \forall z, (x \to z \wedge y \to z) \Rightarrow x = y$
- *surjectivity*: $\forall x, \exists y, y \to x$
- decidable on $\mathbb{Z}$ [Amoroso-Patt, 1972], **undecidable** on $\mathbb{Z}^2$ [Kari, 1992]

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Theorem (Folklore)

On $\mathbb{Z}^d$ any injective CA is surjective.

Conjecture [Gottschalk, 1973]

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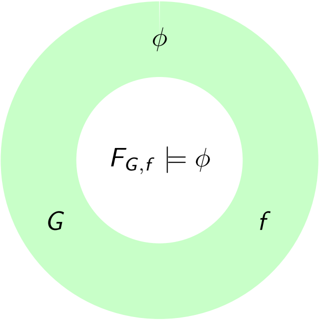
Exercise

Find a CA on a graph G which is injective but not surjective

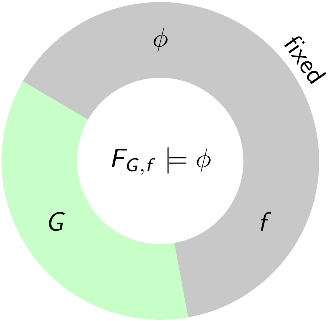
Main theorem 1



Main theorem 1

Set of graphs:

$$\mathcal{L}_{\phi,f} = \{G : F_{G,f} \models \phi\}$$



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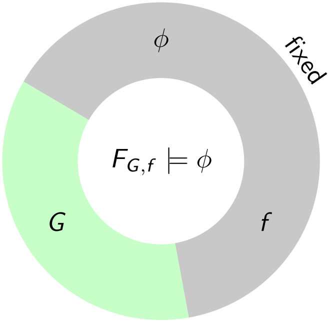

Theorem

If $\mathcal{L}$ is a set of graphs, the following are equivalent:

- $\mathcal{L} = \{G : G \models \Psi\}$ for some **MSO** Ψ,
- $\mathcal{L} = \mathcal{L}_{\phi,f}$ for some **FO** ϕ and **local rule** f.

$+$ effective translations between Ψ and (ϕ, f).

Example: Connected graphs

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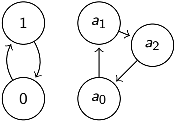

$$f(L) = \begin{cases} 1 - L(\epsilon) & \text{if } L(\epsilon) \in \{0, 1\} \\ a_{i+1 \bmod 3} & \text{if } L(\epsilon) = a_i \text{ and } \{0, 1\} \cap L(u) = \emptyset \\ 0 & \text{else.} \end{cases}$$

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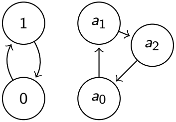

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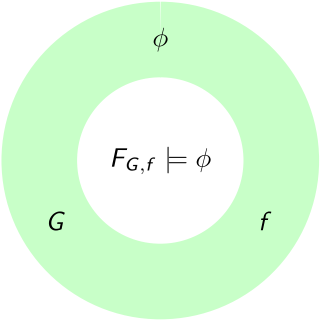
Claim

G connected **IFF** $F_{G,f}$ has no periodic orbit of minimal period 6.

Main theorem 2



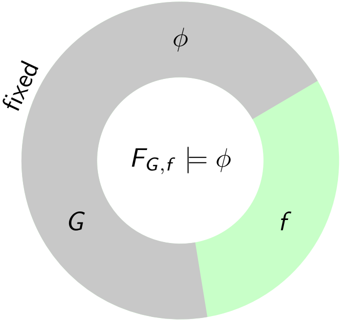
Main theorem 2

Set of local rules:

$$\mathcal{F}_{G,\phi} = \{f : F_{G,f} \models \phi\}$$

Model checking of ϕ on G:

- *input*: local rule f
- *question*: $f \in \mathcal{F}_{G,\phi}$?



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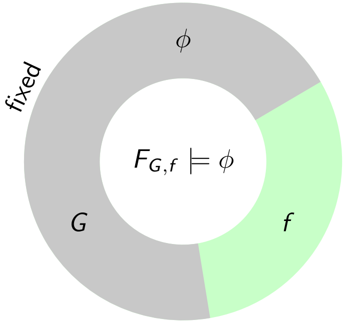

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Theorem

$\forall$ MSO formula Ψ, $\exists$ ϕ and f s.t. $\forall$ **connected** graph G:

$$G \in \mathcal{L}_\Psi \Leftrightarrow G \in \mathcal{L}_{\phi,f}.$$

ϕ only depends on the **prefix signature** of Ψ.

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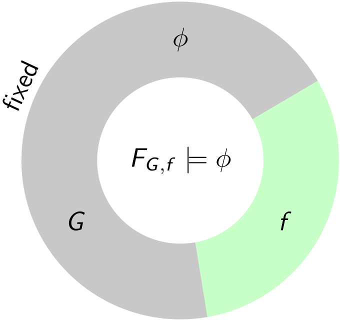

Model checking of ϕ on G:

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Corollary

G connected, $\mathcal{F}$ fragment of MSO of fixed **prefix signature**. Then there is ϕ s.t.

$$\mathcal{F} \leq_m \mathcal{F}_{G,\phi}$$

$\mathcal{F}$ undecidable on $G \Rightarrow$ model checking of ϕ undecidable on G.

More corollaries

Corollary (model checking)

$\exists$ FO formula ϕ such that $\forall$ G connected bounded-degree

ϕ model checking decidable on G
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Ballier-Stein conjecture

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Corollary (finite satisfiability)

$\exists$ FO formula ϕ such that the following problem is **undecidable**:

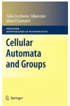
- *input*: local rule f
- *question*: is there some finite G with $F_{G,f} \models \phi$

FO logic on CA orbits = MSO logic

- read the paper ☺
  - slightly more general def of CA
  - other corollaries (non-arithmetical lower bounds)
  - extension of FO and variants of domino problem
  - short-term research directions

- *broadening Ballier-Stein conjecture*
  - how $\phi \mapsto$ Turing-degree($\mathcal{F}_{\phi,G}$) depends on G?
  - beyond Cayley graphs + FO extension
  - NB: $\exists$ 4-regular graph with decidable domino problem but undecidable MSO

- *broadening Gottschalk conjecture*
  - what are the FO tautologies? how do they depend on G?
  - same in FO extension ($\rightarrow$ Garden-of-Eden Theorem)



- "*Cellular automata and groups*" Ceccherini-Silberstein & Coornaert

A word about the proof

MSO

$\exists X_1, \forall x_2, \exists X_3, R(X_1, x_2 X_3)$

FO + local rule

$$\forall c_1, \exists c_2, \forall c_3, \qquad \begin{array}{l} \circlearrowright \\ c_1 \sim X_1 \\ \uparrow \neq \\ c_2 \sim (X_1, x_2) \\ \uparrow \neq \\ c_3 \sim (X_1, x_2, X_3) \end{array}$$

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  1. first-order variable assignment $\sim$ configurations with a single 1
  2. checking $R(X_1, x_2, X_3)$ by local rules when in configuration c_3
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- **solutions:** two "sub-routines" combining FO + local rules
  - *leader election* (configurations with a single 1)
  - *agreement* (uniform configurations)