On the Dynamics of Bounded-Degree Automata Networks

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Automata Networks

abstract definition

- Q : alphabet
- n : number of components
- Q^n : space of configurations
- $F : Q^n \to Q^n$: global map (finite dynamical system)
- orbits: $x, F(x), F^2(x), \ldots$

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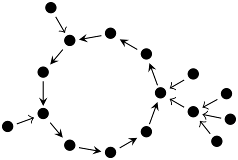

- graph of **dynamics** $G_{\mathrm{dyn}} = (Q^n, \{(x, F(x)) : x \in Q^n\})$
- considered up to isomorphism

Automata Networks

concrete definition

- $G_{\mathtt{com}} = (V, E)$ a **communication** graph
- $|V| = n$
- local maps: $\delta_v : Q^{N^-(v)} \to Q$
- global map: $F : Q^V \to Q^V$ such that

$$F(x)_v = \delta_v(x_{|N^-(v)})$$

- *minimal communication graph = interaction graph*

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What are the possible $G_{\mathtt{dyn}}$ when constraining $G_{\mathtt{com}}$?

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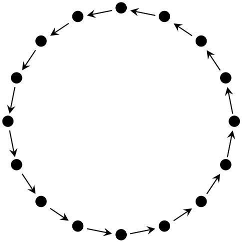
General question

What are the possible G_{dyn} when constraining G_{com}?

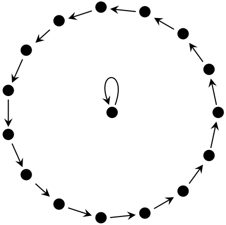
- **this talk:** G_{com} is of **bounded degree** (wrt n).

3 examples of $G_{\mathtt{dyn}}$



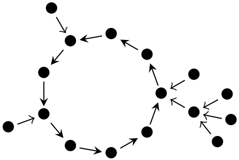
1 cycle

2^n



1 cycle + 1 fixed point

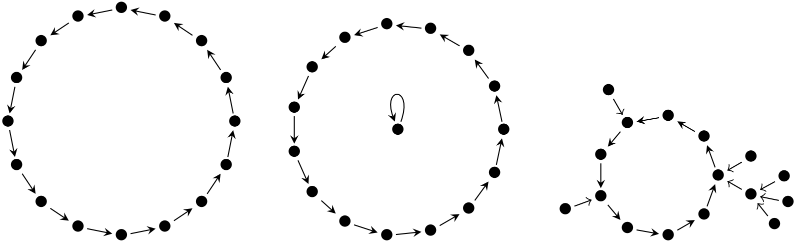
$(2^n - 1) + 1$



1 cycle + constant size tree

$(2^n - C) + C$

3 examples of $G_{\texttt{dyn}}$



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Question

Which one can be realized with bounded degree $G_{\texttt{com}}$?

Bounded degree dynamics

Impossibility results

- fix some degree d
- $q = |Q|$
- $\mathcal{F}(n, q, d)$: maps over Q^n with $G_{\mathtt{com}}$ of degree $\leq d$

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Proposition

If $F \in \mathcal{F}(n, q, d)$ is not the identity, then it has at most $q^n - q^{n-d}$ fixed points.

Theorem

If $F \in \mathcal{F}(n, q, d)$ is not bijective, then its rank is at most $q^n - \frac{n}{d+1}$.

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- **key proof ingredient:** $k-$balance

Bounded degree dynamics

Complexity of recognition

- problem **BDD**
  - d is any fixed *parameter*
  - *input:* G_{dyn} given by Boolean circuits describing map F
  - *question:* can G_{dyn} be realized by G_{com} of degree $\leq d$?

Bounded degree dynamics

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Theorem

BDD is PSPACE and co-NP-hard.

- ⚠ dynamics are up to isomorphism
- without isomorphism, we get a co-NP-complete problem

Question

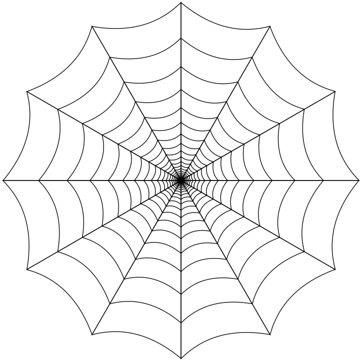
Is **BDD** NP-hard? higher in the polynomial hierarchy?

Bounded degree dynamics

Global picture

Bounded degree dynamics

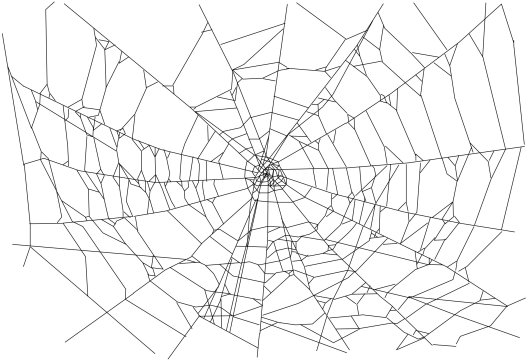
Global picture



- sparse

Bounded degree dynamics

Global picture



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Bounded degree dynamics

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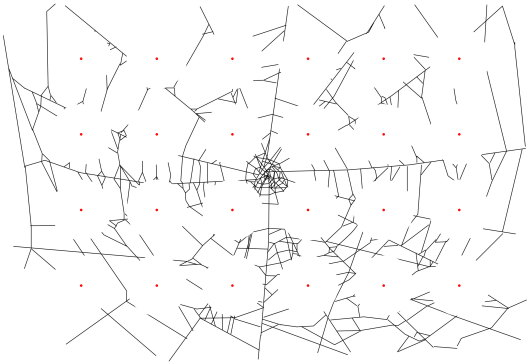

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Bounded degree dynamics

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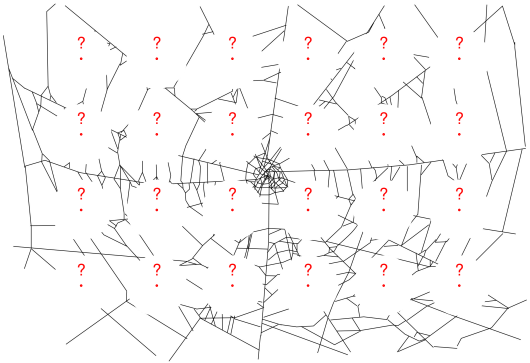

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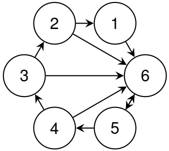
- **what bijections can be realized?**

Realization results

- **Feedback shift registers**
- $g : \{0,1\}^n \to \{0,1\}$
- $F_g(x_1, \ldots, x_n) = (x_2, \ldots, x_n, g(x))$
- "*almost degree 1*"



$G_{\mathtt{com}}$

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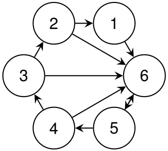
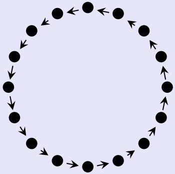
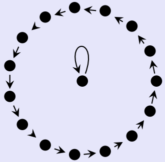


$G_{\mathtt{com}}$

Proposition



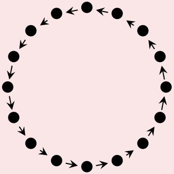
by FSR of degree n



by FSR of degree n or
LFSR of degree 2 for some n

Work in progress

Aracena's conjecture

 cannot be realized with bounded degree $G_{\mathtt{com}}$.

Unpublished theorem (Bridoux-Richard)

For **FSR**, the above G_{dyn} requires degree n.