Expansive Automata Networks

Journées SDA2 2019

F. Bridoux, M. Gadouleau, *G. Theyssier*

Institut de mathématiques de Marseille
(CNRS, Université Aix-Marseille)

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Automata Networks

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- dynamically: all orbits are ultimately periodic

- yes but Q^n has some structure!
- $x = (x_1, \dots, x_n)$

Dependency Graph

- given $F : Q^n \to Q^n$
- define digraph $G_F = (\{1, \cdots, n\}, E)$ by

$$(i, j) \in E \Leftrightarrow \begin{cases} \exists x, y : F(x)_i \neq F(y)_i \\ \text{with } x \text{ and } y \text{ differing only at coordinate } j \end{cases}$$

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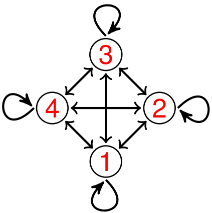
- examples:

  - $F = 0 \cdots 0 \leftrightarrow 1 \cdots 1$
  - $G_F =$



  - $F(x)_k = x_{k+1 \bmod n}$
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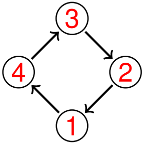

Automata Networks Theory

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Robert' Theorem

If G_F is acyclic then F is nilpotent (F^n is constant).

Feedback bound

$$\left|\{x : F(x) = x\}\right| \leq |Q|^{\nu(G_F)}$$

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Thomas' first rule

If G_F has no positive cycle then F has at most one fixed point.

Positive feedback bound

$\big|\{x : F(x) = x\}\big| \leq |Q|^{\nu^+(G_F)}$

Classical Expansiveness

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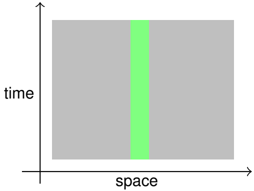

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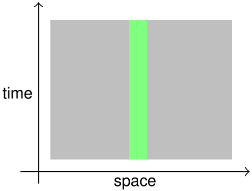

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- observability in automata networks

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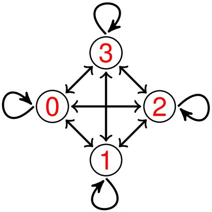

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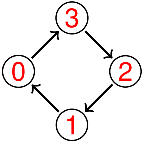

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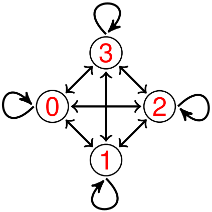

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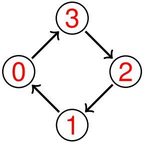

- variations on the definition

Expansion time

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1. for any expansive F: $n \leq T(F) \leq |Q|^n$
2. $(\forall n)$ there is F with $T(F) = |Q|^n - |Q| - 1$

- twisted lexicographic order:

$$00 \to 01 \to 02 \to 10 \to 11 \to 12 \to 20 \to 22 \to 21 \to 00$$

Linear networks

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- Then τ_i^n is a linear bijective map:

$$\tau_i^n = \begin{pmatrix} M_{1,i} & \cdots & M_{n,i} \\ \vdots & \cdots & \vdots \\ M_{1,i}^n & \cdots & M_{n,i}^n \end{pmatrix}$$

  where $F^t = (M_{i,j}^t)$
- for F linear:

F expansive $\Leftrightarrow$ $det(\tau_i^n) \neq 0$ for all i

Graphs allowing expansiveness

- for which G is there F expansive with $G_F = G$?

$$\begin{tikzcd} 0 \arrow[loop left] \arrow[r, leftrightarrow] & 1 \arrow[r, leftrightarrow] & 2 \\ & 3 & \end{tikzcd}$$

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- robust to slight variations in the definition of expansiveness

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$P \in \mathbb{F}_q[X_1, \ldots, X_k]$, non-zero, total degree d, then:

$$Pr(P(a_1, \ldots, a_k) = 0) \leq \frac{d}{|\mathbb{F}_q|}$$

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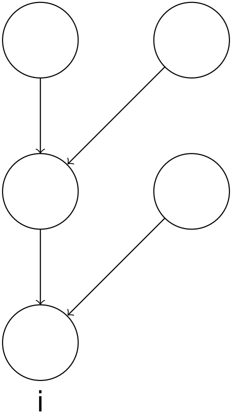
$$det(\tau_i^n) = \sum_{\sigma}(-1)^{s(\sigma)} \prod_{t=1}^{n} (\tau_i^n)_{\sigma(t),t}$$



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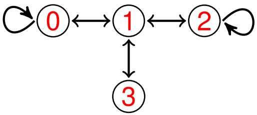

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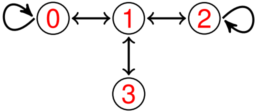

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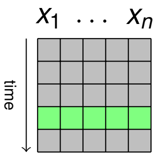
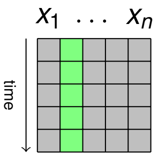
Open

Upper-bound on Q for all admissible graphs of fixed degree d?

Super-expansiveness



bijective



expansive in time *n*

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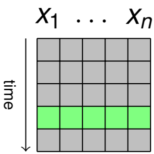

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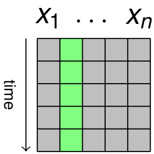

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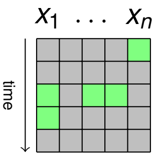

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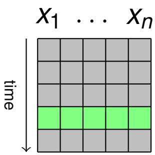

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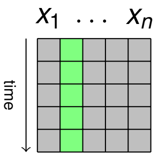

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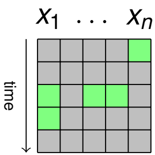

- $O = \big((i_1, t_1), \dots, (i_n, t_n)\big)$
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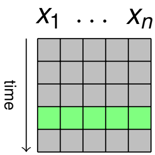

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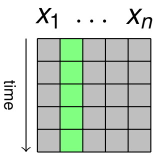

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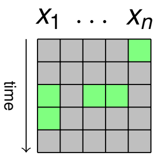

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Definition

F super-expansive if τ_O injective for any O of length n.

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- Singleton bound: $\#\text{words} \leq q^{\text{length}-\text{distance}+1}$
- In our case: $q^n \leq q^{n^2-(n^2-n+1)+1}$

Going further

- expansion frequency
- block-sequential update schedules
- link with other “topological” properties
- observability in general