Revisiting Circuit Embeddings in Cellular Automata

Computation and Dynamics Workshop, Toulouse

G. Theyssier

(joint work with E. Goles, P. Montealegre, K. Perrot)

Institut de Mathématiques de Marseille (CNRS, AMU)

26 mars 2018

Setting of this talk

- **configurations:** $Q^{\mathbb{Z}^D}$, Q finite
- **topology:**

$$d(x,y) = 2^{-\min\{\|z\| \in \mathbb{Z}^D : x_z \neq y_z\}}$$

- **cylinder** associated to $u \in Q^M$, with $M \subseteq \mathbb{Z}^D$ finite

$$[u] = \{c \in Q^{\mathbb{Z}^D} : c|_M = u\}$$

- **cellular automaton:** $F : Q^{\mathbb{Z}^D} \to Q^{\mathbb{Z}^D}$

$$\forall z \in \mathbb{Z}^D, F(c)_z = f\big(j \in V \mapsto c_{z+j}\big)$$

$V \subseteq \mathbb{Z}^D$ finite **neighborhood**, $f : Q^V \to Q$ **local rule**

- $D = 2$

General Motivation

Cellular automata claimed universal/computationally hard via Boolean circuit embedding:

- Banks' rule (Bank, 1970s)
- Game of Life (Conway, 1970s)
- Abelian sandpile 3D (Moore-Nilsson, 1990s)
- Majority vote 3D + Ising Dynamics (Moore, 1990s)
- Lattice gases (Moore-Nordhal, 1990s)
- Life without death (Griffeath-Moore, 1990s)
- etc

General Motivation

Cellular automata claimed universal/computationally hard via Boolean circuit embedding:

- Banks' rule (Bank, 1970s)
- Game of Life (Conway, 1970s)
- Abelian sandpile 3D (Moore-Nilsson, 1990s)
- Majority vote 3D + Ising Dynamics (Moore, 1990s)
- Lattice gases (Moore-Nordhal, 1990s)
- Life without death (Griffeath-Moore, 1990s)
- etc

Problems

1. are they all computationally equivalent?
2. can we deduce dynamical feature Y is hard from the fact that feature X is hard?

General Motivation

Cellular automata claimed universal/computationally hard via Boolean circuit embedding:

- Banks' rule (Bank, 1970s)
- Game of Life (Conway, 1970s)
- Abelian sandpile 3D (Moore-Nilsson, 1990s)
- Majority vote 3D + Ising Dynamics (Moore, 1990s)
- Lattice gases (Moore-Nordhal, 1990s)
- Life without death (Griffeath-Moore, 1990s)
- etc

Problems

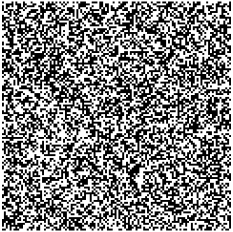
1. are they all computationally equivalent?
2. can we deduce dynamical feature Y is hard from the fact that feature X is hard?

- **this talk:** focus on embeddings

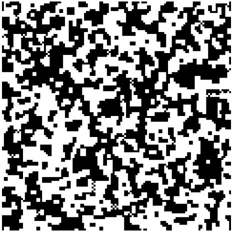
Concrete motivation

- $Q = \{0, 1\}$
- V unit ball of $\mathbb{Z}^2$ for $\| \cdot \|_1$ (von Neumann neighb.)

$$F(x)_z = \text{majority}(\{x_{z+i} : i \in V\})$$



$t = 0$

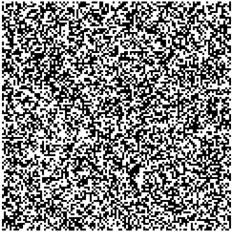


$t = 500$

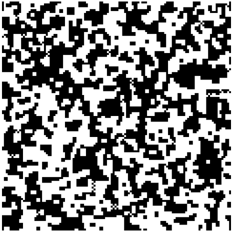
Concrete motivation

- $Q = \{0, 1\}$
- V unit ball of $\mathbb{Z}^2$ for $\|\cdot\|_1$ (von Neumann neighb.)

$$F(x)_z = \text{majority}(\{x_{z+i} : i \in V\})$$



$t = 0$



$t = 500$

Questions

- can this rule compute?
- is it resistant to noise?

Dynamical Questions Considered

- **prediction** problem

  input: c finite, t, z with $|c| = (2rt+1)^D$

  question: what is $F^t(c)_z$?

Dynamical Questions Considered

- **prediction** problem

  input: c finite, t, z with $|c| = (2rt + 1)^D$

  question: what is $F^t(c)_z$?

- **reachability** problem

  input: c eventually periodic, u

  question: is there t such that $F^t(c) \in [u]$

Dynamical Questions Considered

- **prediction** problem

  input: c finite, t, z with $|c| = (2rt+1)^D$
  question: what is $F^t(c)_z$?

- **reachability** problem

  input: c eventually periodic, u
  question: is there t such that $F^t(c) \in [u]$

- f**-cycle** problem

  input: c (spatially) periodic of period $n \times \cdots \times n$
  question: is there $k \leq f(n)$ and t with $F^{t+k}(c) = F^t(c)$?

Dynamical Questions Considered

- **prediction** problem

  input: c finite, t, z with $|c| = (2rt+1)^D$
  question: what is $F^t(c)_z$?

- **reachability** problem

  input: c eventually periodic, u
  question: is there t such that $F^t(c) \in [u]$

- *f*-**cycle** problem

  input: c (spatially) periodic of period $n \times \cdots \times n$
  question: is there $k \leq f(n)$ and t with $F^{t+k}(c) = F^t(c)$?

- etc

Complexity Cheat Sheet

Complexity Cheat Sheet

- LOGSPACE $\subseteq$ NC $\subseteq$ P $\subseteq$ NP $\subseteq \cdots \subseteq$ PSPACE

Complexity Cheat Sheet

- LOGSPACE $\subseteq$ NC $\subseteq$ P $\subseteq$ NP $\subseteq \cdots \subseteq$ PSPACE
- PSPACE-completeness, P-completeness, NC-completeness

Complexity Cheat Sheet

- LOGSPACE $\subseteq$ NC $\subseteq$ P $\subseteq$ NP $\subseteq \cdots \subseteq$ PSPACE
- PSPACE-completeness, P-completeness, NC-completeness
- Communication complexity (CC)
  - $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$

Complexity Cheat Sheet

- LOGSPACE $\subseteq$ NC $\subseteq$ P $\subseteq$ NP $\subseteq \cdots \subseteq$ PSPACE
- PSPACE-completeness, P-completeness, NC-completeness
- Communication complexity (CC)
  - $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$
  - Alice knows x (not y), Bob knows y (not x), both know f
  - they must communicate to decide $f(x,y)$

Complexity Cheat Sheet

- LOGSPACE $\subseteq$ NC $\subseteq$ P $\subseteq$ NP $\subseteq \cdots \subseteq$ PSPACE
- PSPACE-completeness, P-completeness, NC-completeness
- Communication complexity (CC)
  - $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$
  - Alice knows x (not y), Bob knows y (not x), both know f
  - they must communicate to decide $f(x,y)$
  - $CC(f)$ how many bits do they have to exchange in the worst case

Complexity Cheat Sheet

- LOGSPACE $\subseteq$ NC $\subseteq$ P $\subseteq$ NP $\subseteq \cdots \subseteq$ PSPACE
- PSPACE-completeness, P-completeness, NC-completeness
- Communication complexity (CC)
  - $f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}$
  - Alice knows x (not y), Bob knows y (not x), both know f
  - they must communicate to decide $f(x,y)$
  - $CC(f)$ how many bits do they have to exchange in the worst case

f	CC(f)
$\sum_i x_i + y_i$ mod 2	1
$\sum_i x_i y_i$ mod 2	n

Complexity Cheat Sheet

- LOGSPACE $\subseteq$ NC $\subseteq$ P $\subseteq$ NP $\subseteq \cdots \subseteq$ PSPACE
- PSPACE-completeness, P-completeness, NC-completeness
- Communication complexity (CC)
  - $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$
  - Alice knows x (not y), Bob knows y (not x), both know f
  - they must communicate to decide $f(x,y)$
  - $CC(f)$ how many bits do they have to exchange in the worst case

f	CC(f)
$\sum_i x_i + y_i$ mod 2	1
$\sum_i x_i y_i$ mod 2	n

  - applications (VLSI, time complexity, circuit complexity, etc)

Dynamical Questions

	Prediction	Reachability	f-cycle
General	P-complete $CC \in \Omega(n^D)$	undecidable	PSPACE-hard

Dynamical Questions

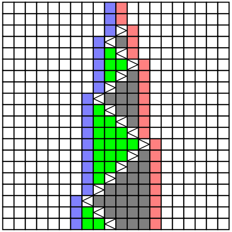
	Prediction	Reachability	f-cycle
General	P-complete $CC \in \Omega(n^D)$	undecidable	PSPACE-hard
Example 1	LOGSPACE $CC \in O(\log(n))$	undecidable	trivial

Dynamical Questions

	Prediction	Reachability	f-cycle
General	P-complete $CC \in \Omega(n^D)$	undecidable	PSPACE-hard
Example 1	LOGSPACE $CC \in O(\log(n))$	undecidable	trivial
Example 2	P-complete $CC \in \Omega(\sqrt{n})$	decidable	trivial

Dynamical Questions

	Prediction	Reachability	f-cycle
General	P-complete $CC \in \Omega(n^D)$	undecidable	PSPACE-hard
Example 1	LOGSPACE $CC \in O(\log(n))$	undecidable	trivial
Example 2	P-complete $CC \in \Omega(\sqrt{n})$	decidable	trivial

- Ex. 1: Minsky machine in 1D freezing CA ($F(c)_z \leq c_z$)
- Ex. 2: 

Not Any Dynamical Question

Sub-system $\perp$ Factor

Not Any Dynamical Question

Sub-system $\perp$ Factor

- Limit set
- Blondel-Delevenne-Kůrka universality

Not Any Dynamical Question

Sub-system $\perp$ Factor

- Limit set
- Blondel-Delevenne-Kůrka universality
  - **column factor** = orbits with precision ϵ
  - **model checking** = properties defined by finite automaton

    ex: *can you reach* $[u]$ *from* $[v]$ *passing through* $[w]$*?*
  - universal = model checking undecidable on column factor

Not Any Dynamical Question

Sub-system $\perp$ Factor

- Limit set
- Blondel-Delevenne-Kůrka universality
  - **column factor** = orbits with precision ϵ
  - **model checking** = properties defined by finite automaton

    ex: *can you reach* $[u]$ *from* $[v]$ *passing through* $[w]$*?*
  - universal = model checking undecidable on column factor

Theorem (Guillon-Meunier-T.,2010)

For any CA F over alphabet Q, we can build G and H with:

- $G|_{Q^{\mathbb{Z}^D}} = H|_{Q^{\mathbb{Z}^D}} = F$
- column factors of G are SFT of order 2
- H has a limit set with NLOGSPACE computable language

Not Any Dynamical Question

Sub-system $\perp$ Factor

- Limit set
- Blondel-Delevenne-Kůrka universality
  - **column factor** = orbits with precision ϵ
  - **model checking** = properties defined by finite automaton

    ex: *can you reach* $[u]$ *from* $[v]$ *passing through* $[w]$*?*
  - universal = model checking undecidable on column factor

Theorem (Guillon-Meunier-T.,2010)

For any CA F over alphabet Q, we can build G and H with:

- $G|_{Q^{\mathbb{Z}^D}} = H|_{Q^{\mathbb{Z}^D}} = F$
- column factors of G are SFT of order 2
- H has a limit set with NLOGSPACE computable language

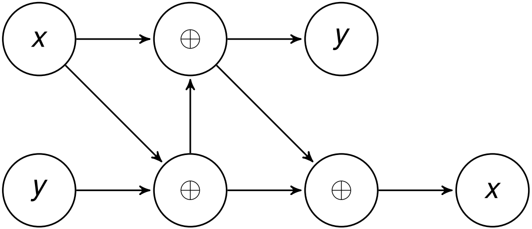
Question ($D = 1$)

Can a positively expansive CA compute?

Boolean Circuits

Boolean Circuits

- **definition:** a DAG with a Boolean map $f_v : \{0,1\}^2 \to \{0,1\}$ on each node v
- circuit execution:
  - nodes with in-degree 0 = layer 1 = **inputs**
  - layer $n+1$: apply f_v on values of layer $\leq n$
  - nodes with out-degree 0 = **outputs**
- induces a map $F : \{0,1\}^n \to \{0,1\}^p$
- example:



Planar + Monotone

Planar + Monotone

- $\mathbb{Z}^2$
- majority rule (like many interesting examples) is monotone

Planar + Monotone

- $\mathbb{Z}^2$
- majority rule (like many interesting examples) is monotone

Folklore

- planar circuit are equally powerful as general circuits
- monotone circuit are equally powerful as general circuits

Planar + Monotone

- $\mathbb{Z}^2$
- majority rule (like many interesting examples) is monotone

Folklore

- planar circuit are equally powerful as general circuits
- monotone circuit are equally powerful as general circuits

Theorem (Goldschlager, 1980 + Barrington *et al.*, 1999)

A planar monotone Boolean circuit of polynomial size can be simulated by a circuit of polynomial size and polylog depth (hence in NC).

Planar + Monotone

- $\mathbb{Z}^2$
- majority rule (like many interesting examples) is monotone

Folklore

- planar circuit are equally powerful as general circuits
- monotone circuit are equally powerful as general circuits

Theorem (Goldschlager, 1980 + Barrington *et al.*, 1999)

A planar monotone Boolean circuit of polynomial size can be simulated by a circuit of polynomial size and polylog depth (hence in NC).

- conjecture $P \neq NC$
- **key aspect:** information crossing

Modes of Information Propagation

Modes of Information Propagation

$\cdots 0000111 \cdots$

$\leftarrow$

$\cdots 0010000 \cdots$

$\rightarrow$

Embedding Boolean Circuits

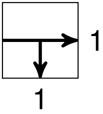
- Agnostic definition

Embedding Boolean Circuits

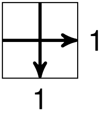
- Agnostic definition
- set $\mathcal{B}$ of $m \times m$ blocks

Embedding Boolean Circuits

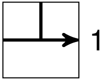
- Agnostic definition
- set $\mathcal{B}$ of $m \times m$ blocks
- for each block $B \in \mathcal{B}$:
  - inputs $I \subseteq \{N, E, S, W\}$, outputs $O \subseteq \{N, E, S, W\}$
  - a map $\phi_B : \{0,1\}^I \to \{0,1\}^O$
  - output value $\nu_B \in Im(\phi_B)$



$\phi_B(x) = (x, x)$



$\phi_B(x, y) = (x, y)$

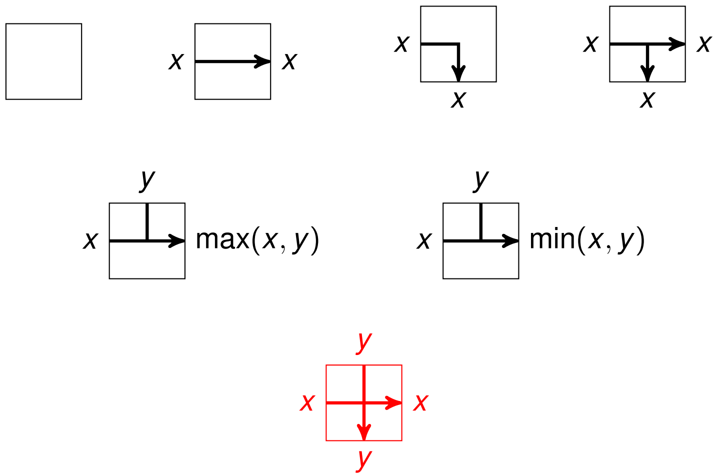


$\phi_B(x, y) = \max(x, y)$

Circuitery

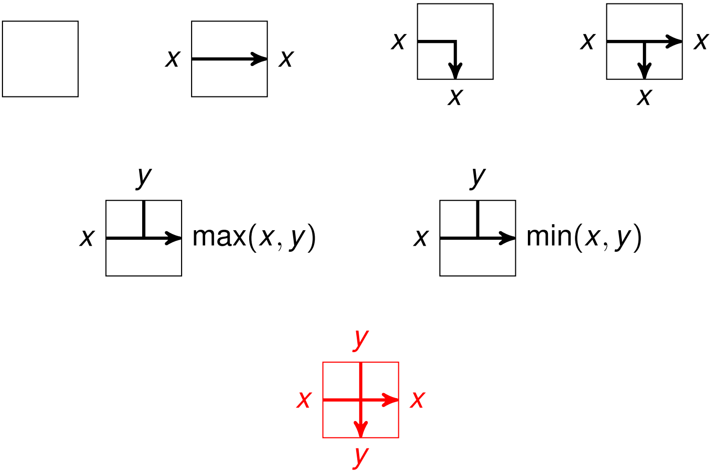
Circuitery

- up to rotation/reflexions:



Circuitery

- up to rotation/reflexions:



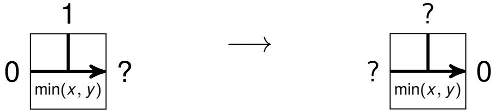
- **Remark:** $\phi(0, \cdots, 0) = (0, \cdots, 0)$

Two Modes of Simulation

- **valid configuration:** concatenation of blocks of $\mathcal{B}$ with output/input matching on edges

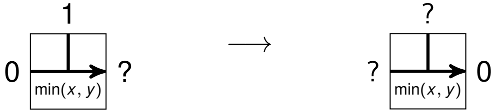
Two Modes of Simulation

- **valid configuration:** concatenation of blocks of $\mathcal{B}$ with output/input matching on edges
- **block update:** ϕ unchanged, values updated according to input neighbors



Two Modes of Simulation

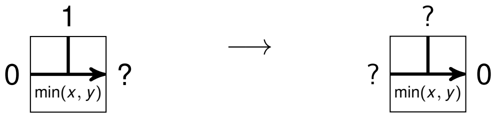
- **valid configuration:** concatenation of blocks of $\mathcal{B}$ with output/input matching on edges
- **block update:** ϕ unchanged, values updated according to input neighbors



- **strong simulation:** $\exists\Delta, \forall c$ valid, $F^{\Delta}(c)$ valid, all blocks updated

Two Modes of Simulation

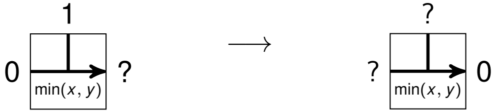
- **valid configuration:** concatenation of blocks of $\mathcal{B}$ with output/input matching on edges
- **block update:** ϕ unchanged, values updated according to input neighbors



- **strong simulation:** $\exists\Delta, \forall c$ valid, $F^{\Delta}(c)$ valid, all blocks updated
- **weak simulation:** idem but
  - active/inactive blocks
  - update on active block when $1 \in$ inputs, then inactive
  - no update on inactive blocks

Two Modes of Simulation

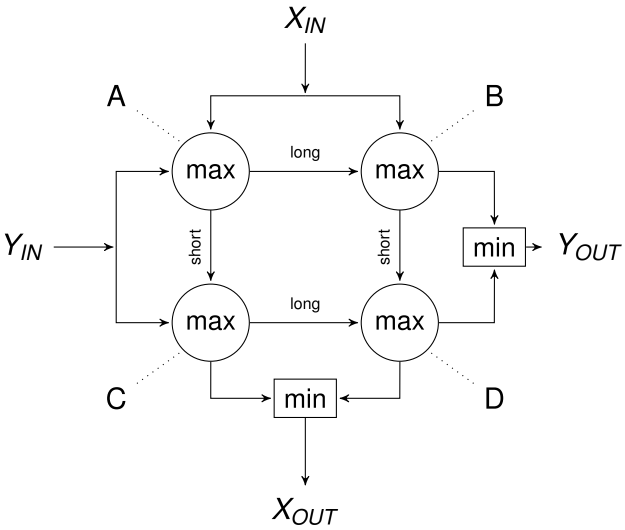
- **valid configuration:** concatenation of blocks of $\mathcal{B}$ with output/input matching on edges
- **block update:** ϕ unchanged, values updated according to input neighbors



- **strong simulation:** $\exists \Delta, \forall c$ valid, $F^{\Delta}(c)$ valid, all blocks updated
- **weak simulation:** idem but
  - active/inactive blocks
  - update on active block when $1 \in$ inputs, then inactive
  - no update on inactive blocks
- *(we could imagine weaker (unstable) simulations)*

Crossings for Strong Simulations

Crossings for Strong Simulations



Intrinsic Universality

- simulation of CA G by CA F:
  - state $q \leftrightarrow$ set of $m \times m$ blocks $\mathcal{B}_q$
  - 1 step $\leftrightarrow \Delta$ steps

Intrinsic Universality

- simulation of CA G by CA F:
  - state $q \leftrightarrow$ set of $m \times m$ blocks $\mathcal{B}_q$
  - 1 step $\leftrightarrow$ Δ steps
- F **intrinsically universal** if it can simulate any G

Intrinsic Universality

- simulation of CA G by CA F:
  - state $q \leftrightarrow$ set of $m \times m$ blocks $\mathcal{B}_q$
  - 1 step $\leftrightarrow \Delta$ steps
- F **intrinsically universal** if it can simulate any G

Theorem

F can strongly simulate monotone circuitery **iff** it is intrinsically universal.
In this case:

- prediction is P-complete
- CC(prediction) $\in \Omega(n^2)$
- reachability is undecidable
- f-cycle is PSPACE-hard

Intrinsic Universality

- simulation of CA G by CA F:
  - state $q \leftrightarrow$ set of $m \times m$ blocks $\mathcal{B}_q$
  - 1 step $\leftrightarrow$ Δ steps
- F **intrinsically universal** if it can simulate any G

Theorem

F can strongly simulate monotone circuitery **iff** it is intrinsically universal.
In this case:

- prediction is P-complete
- CC(prediction) $\in \Omega(n^2)$
- reachability is undecidable
- f-cycle is PSPACE-hard

Weak Universality

Weak Universality

Theorem

If F weakly simulate a monotone circuitery + crossing, then:

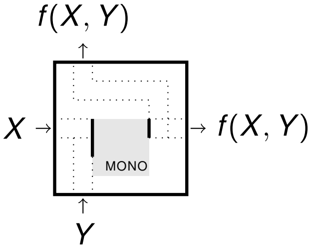
- it can simulate any 1D CA on diagonals
- prediction is P-complete
- CC(prediction) $\in \Omega(n \log(n))$
- reachability is undecidable

Weak Universality

Theorem

If F weakly simulate a monotone circuitery + crossing, then:

- it can simulate any 1D CA on diagonals
- prediction is P-complete
- CC(prediction) $\in \Omega(n \log(n))$
- reachability is undecidable

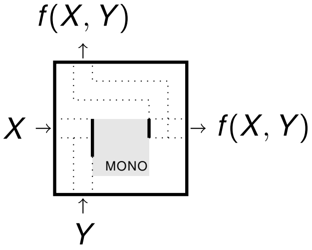


Weak Universality

Theorem

If F weakly simulate a monotone circuitery + crossing, then:

- it can simulate any 1D CA on diagonals
- prediction is P-complete
- CC(prediction) $\in \Omega(n \log(n))$
- reachability is undecidable



- examples with CC $\in O(n \log(n))$ and trivial f-cycle pb.

Application: weighted majority

- $Q = \{-1, 1\}$, V von Neumann
- weight $w : V \to Q$
- symmetric weight: $w(-z) = w(z)$

Application: weighted majority

- $Q = \{-1, 1\}$, V von Neumann
- weight $w : V \to Q$
- symmetric weight: $w(-z) = w(z)$
- **uniform** weighted majority (w fixed):

$$F_w(x)_z = \begin{cases} 1 & \text{if } \sum_{i \in V} w(i) x_{z+i} > 0, \\ -1 & \text{else.} \end{cases}$$

Application: weighted majority

- $Q = \{-1, 1\}$, V von Neumann
- weight $w : V \to Q$
- symmetric weight: $w(-z) = w(z)$
- **uniform** weighted majority (w fixed):

$$F_w(x)_z = \begin{cases} 1 & \text{if } \sum_{i \in V} w(i) x_{z+i} > 0, \\ -1 & \text{else.} \end{cases}$$

- **non-uniform** weighted majority (w read in configuration):

$$F : (Q^6)^{\mathbb{Z}^2} \to (Q^6)^{\mathbb{Z}^2}$$

Application: weighted majority

- $Q = \{-1, 1\}$, V von Neumann
- weight $w : V \to Q$
- symmetric weight: $w(-z) = w(z)$
- **uniform** weighted majority (w fixed):

$$F_w(x)_z = \begin{cases} 1 & \text{if } \sum_{i \in V} w(i) x_{z+i} > 0, \\ -1 & \text{else.} \end{cases}$$

- **non-uniform** weighted majority (w read in configuration):

$$F : (Q^6)^{\mathbb{Z}^2} \to (Q^6)^{\mathbb{Z}^2}$$

- **non-uniform symmetric** weighted majority:

$$w'_z(i) = w_z(i) w_{z+i}(-i)$$

Application: weighted majority

Theorem

The following circuit simulations hold:

	Symmetric	Asymmetric
Uniform	weak	weak + cross
Non-uniform	weak + cross	strong

Application: weighted majority

Theorem

The following circuit simulations hold:

	Symmetric	Asymmetric
Uniform	weak	weak + cross
Non-uniform	weak + cross	strong

Theorem (Goles-Olivos, 1980)

Symmetric rules have trivial f-cycle problem.

Application: weighted majority

Theorem

The following circuit simulations hold:

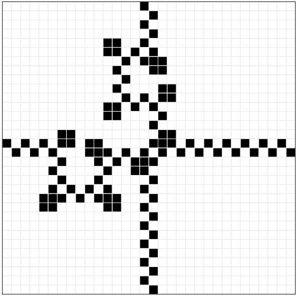
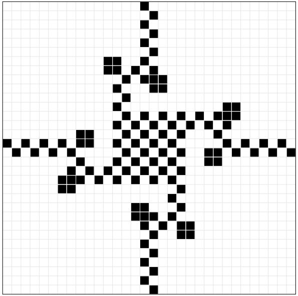
	Symmetric	Asymmetric
Uniform	weak	weak + cross
Non-uniform	weak + cross	strong

Theorem (Goles-Olivos, 1980)

Symmetric rules have trivial f-cycle problem.

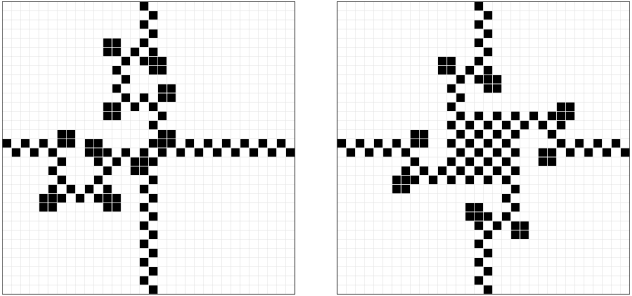
- **Corollary:** strong simulation impossible for symmetric rules

Back to Majority

- weak monotone simulation
- best lower bound: prediction NC^1-hard

Back to Majority



- weak monotone simulation
- best lower bound: prediction NC^1-hard
- *communication complexity in* $o(n \log(n))$*?*