On Cold Universality in Cellular Automata

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Definitions

- **configurations:** $Q^{\mathbb{Z}^D}$, Q finite

- **topology:**

$$d(x,y) = 2^{-\min\{\|z\| \in \mathbb{Z}^D : x_z \neq y_z\}}$$

- **cellular automaton:** $F : Q^{\mathbb{Z}^D} \to Q^{\mathbb{Z}^D}$

$$\forall z \in \mathbb{Z}^D, F(c)_z = f\big(j \in V \mapsto c_{z+j}\big)$$

$V \subseteq \mathbb{Z}^D$ finite **neighborhood**, $f : Q^V \to Q$ **local rule**

Hot dynamics

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Allumer le feu !

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- F is **convergent** if $(F^t(x))_t$ is convergent for all x, *i.e.*

$$\forall x, \exists x_\infty \;:\; d\left(F^t(x), x_\infty\right) \to 0$$

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Question

Is F^2 bounded-change?

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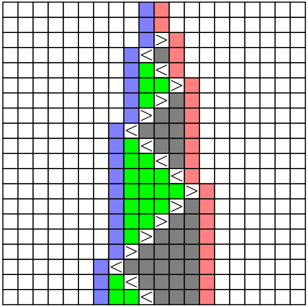

Complexity even in 1D

Complexity even in 1D

Goles, Ollinger, Theyssier

- there is a Turing universal 1D freezing CA
- bounded-change 1D CA have a LOGSPACE prediction
  and $O(\log(n))$ communication complexity
- there are 1D convergent CA with P-complete prediction
  and $\Omega(\sqrt{n})$ communication complexity
- there are convergent 1D CA with non-recursive limit set

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What happens for convergent CA on the free group?

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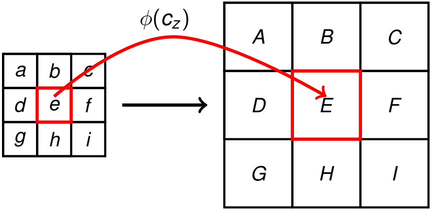
injective



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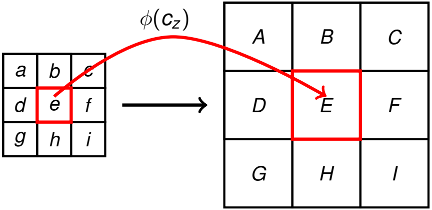
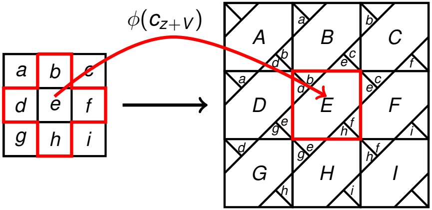


2. **context sensitive:**

   $\phi : Q_G^V \to Q_F^{m \times m}$
   injective global map



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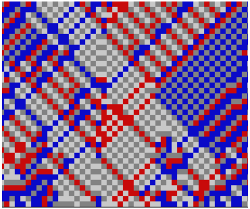

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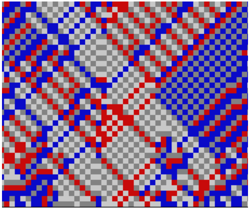

- *F* is **freezing-universal** if it is freezing and can simulate any freezing CA

Freezing-universality

Becker, Maldonado, Ollinger, Theyssier

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No 1-change freezing universal in 2D with von Neumann neighborhood.

Intuition: Jordan's curve lemma

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- synchronization ($\neq$ from aTAM universality)

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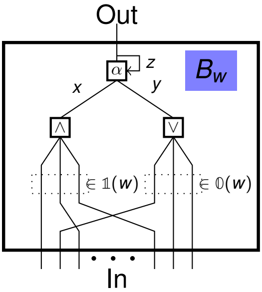

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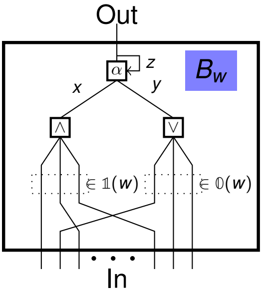

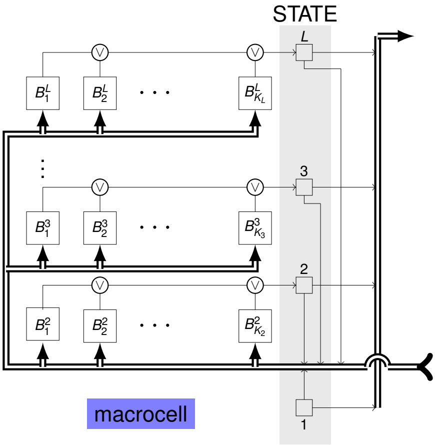

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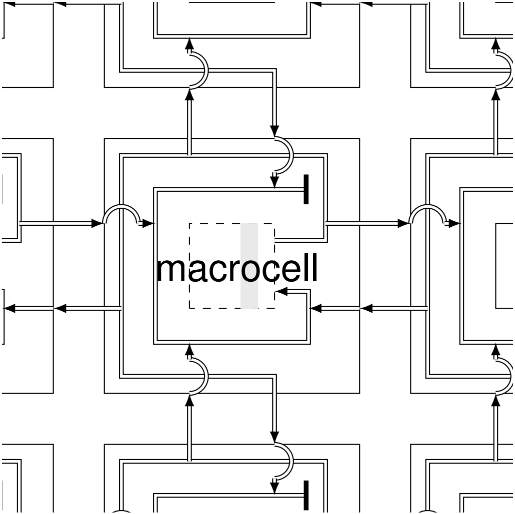

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Theorem

F is (context-sensitively) simulated by some freezing CA
IFF it admits an explicit local energy.

Questions

1. all bounded-change CA have an explicit local energy?
2. is there a bounded-change-universal CA?
3. is there a convergent-universal CA?
4. what are limit sets of bounded-change CAs?