Simulation Preorders in Cellular Automata

From dynamics to computations

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CAUTION
WET
PAINT

What

- **transitive relations (preorders) interpreted as simulations**

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  - *in accordance with the CA model*
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- **must be “natural” and/or “simple”**
  - *in accordance with the CA model*
    - uniformity
    - locality
  - *made of ingredients already present/used in literature*
    - making blocs of cells (higher blockpresentation)
    - shifts, time cutting
    - colouring of states
    - restriction to “natural” and/or “simple” subspaces

Why

- **A transitive relation yields**
  - a notion of equivalence between CA
  - a notion of universality (top of the preorder, possibly empty)
  - a notion of complexity (how high in the preorder)
  - a classification tool (topology of the preorder)
- **formal notions are needed to prove negative results!**

Definitions and examples

The general form of a simulation

Definition

$\mathcal{A} \preceq \mathcal{B}$ if there are

- *a subspace Σ*
- *a coding function f*
- *and parameters d, t, t'*

s.t.

$$\begin{array}{ccc} \Sigma & \xrightarrow{f} & \mathfrak{C}_A \\ {\scriptstyle \sigma_d \circ F_B^t}\big\downarrow & & \big\downarrow{\scriptstyle F_A^{t'}} \\ \Sigma & \xrightarrow{f} & \mathfrak{C}_A \end{array}$$

When $t' = 1$ the simulation is said **total**.

Coding functions

- **A coding function $f : \Sigma \to \mathfrak{C}_A$ must be**
  - onto
  - continuous (Cantor topology)
  - weakly uniform (i.e. weakly commuting with shifts)

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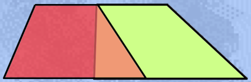
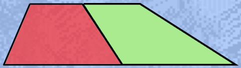
  -  or 

context-sensitive

context-free

Simulation subspaces

- **Σ must be at least**
  - closed (Cantor topology)
  - weakly shift-invariant: $\exists a, \forall x$: $x \in \Sigma \iff \sigma^a(x) \in \Sigma$

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  - sofic subshift:

$$\exists \mathcal{A}, \Sigma_{\mathrm{SFT}} : \Sigma = F_A(\Sigma_{\mathrm{SFT}})$$

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No!

- **we want preorders** (i.e. transitivity)!

$$\begin{array}{ccc} \Sigma & \xrightarrow{f} & \mathfrak{C}_A \\ {\scriptstyle \sigma_d \circ F_B^t}\downarrow & & \downarrow{\scriptstyle F_A^{t'}} \\ \Sigma & \xrightarrow{f} & \mathfrak{C}_A \end{array} \qquad \begin{array}{ccc} \Sigma' & \xrightarrow{f'} & \mathfrak{C}_C \\ {\scriptstyle \sigma_e \circ F_A^u}\downarrow & & \downarrow{\scriptstyle F_C^{u'}} \\ \Sigma' & \xrightarrow{f'} & \mathfrak{C}_C \end{array}$$

$$f^{-1}(\Sigma')?$$

e.g., block + context-sensitive

Selected candidates (for today...)

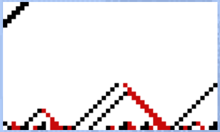
- full space + onto
- block + 1-to-1 + context-free (N.O.'s "groupage")
- block + onto + context-free (N.O.'s + colouring of blocs)
- SFT + sliding + context-sensitive
- sofic + sliding + context-sensitive
- all that is allowed

Proposition

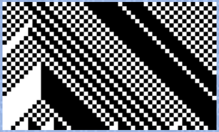
They are all preorders.

Concrete examples

- “184” simulates “just gliders”



$\succeq$



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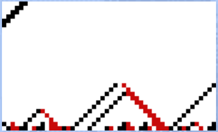
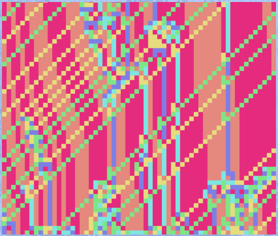
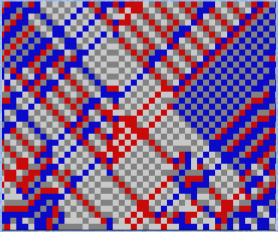


$\preceq$

- two universal CA (for block + context-free simulations)



Nicolas



Gaétan

Periodicity of simulator

- **What if we allow aperiodicity in the simulator?**

Definition (Aperiodic simulation)

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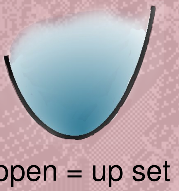
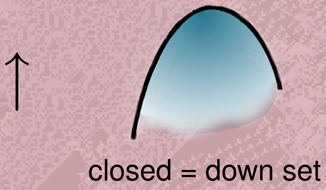
$$

- **Aperiodic simulation can often be made periodic**
  - true for full space or block subspaces simulations,
  - irreductible Σ (i.e. $\exists x \in \Sigma : L(x) = L(\Sigma)$) is a sufficient condition?

Topologies, Decidability, Universality Classes, etc

Order topologies





- some properties are true for all simulation preorders
  - equicontinuous CA = closed set
  - k-heads TM capabilities = open set
  - infinite increasing chains
- some are either true or open
  - reversible CA = closed set
    (open for context-sensitive)
  - non-surjective CA = CA simulating a nilpotent CA
    (open for 1-to-1)
- some are known to depend on the preorder
  - CA sensitive to initial conditions = open set

Universality classes

Definition

$\mathcal{A}$ universal if $\mathcal{B} \preceq \mathcal{A}$ for all $\mathcal{B}$

Proposition

The following simulations have a non empty universal classe:

- *block + 1-to-1 + context-free (N.O.'s "groupage")*
- *block + onto + context-free (N.O.'s + colouring of blocs)*
- *SFT + sliding + context-sensitive*
- *sofic + sliding + context-sensitive*
- *all that is allowed*

- **"universal" simulations**

A useful construction

T **a tile set** $\mapsto \mathcal{A}_T$ **of alphabet** $U \times T \cup \{\kappa\}$

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1D: T must be NE-deterministic

- $\mathcal{A}_T$ plays a universal CA on the U layer;
- $\mathcal{A}_T$ plays T on the T layer;
- T-component of confs are diagonals of T-tilings;
- κ spreading over $U \times T$ if tiling can't be continued

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Let $\mathcal{A}$ be any CA which is not nilpotent over periodic configurations. The following problem is undecidable:

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Proof.

- NE-deterministic T is aperiodic?
- T aperiodic $\Rightarrow \mathcal{A}_T$ nilpotent over periodic confs
- T periodic $\Rightarrow \mathcal{A}_T$ universal

□

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Proof.
There are universal CA which are not nilpotent over periodic confs. □

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- T tiles the plane?
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- $NIL \preceq \mathcal{B} \iff \mathcal{B}$ non-surjective

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YES!

nilpotency is a decidable lower bound for onto simulations

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Theorem

Let $\preceq$ be block + context-free + total.
Let E be any set of CA and E' be its closure by $\times$.
If there is a non-universal $\mathcal{A}$ with $E \preceq \mathcal{A}$ then there is a non-universal $\mathcal{A}'$ with $E' \preceq \mathcal{A}'$.

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- **Open question for block + context-free:**

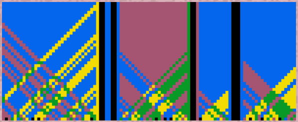
  1-to-1 universal class = onto universal class?

Separating simulation relations

- **1-to-1 vs. onto / block vs. full space**

$\sigma \times \sigma^{-1}$

$$\preceq_{\sqsubseteq} \qquad \not\preceq_{\trianglelefteq}$$



$\sigma \times \sigma^{-1}$ + wall

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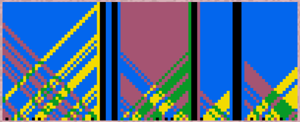

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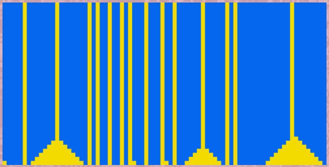

segments reduction

$\not\preceq_{\sqsubseteq}$

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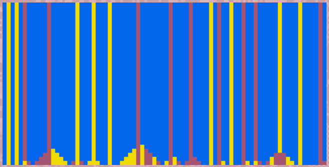

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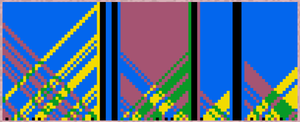

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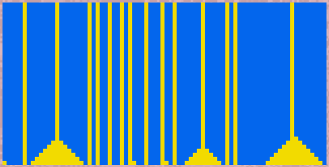

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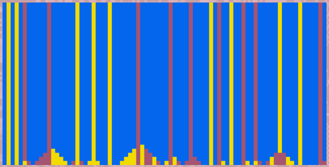

segments reduction with parity

- **context-free + block can't do everything in 1D**

Simulation Preorders as Computation Models?

- **first try**

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Proposition

Let Σ be any SFB and $\mathcal{A}$ be any CA.
If $\mathcal{A}^t(\Sigma) \subseteq \Sigma$ then $(\mathcal{A}, \Sigma)$ is isomorphic to a CA $\mathcal{A}_\Sigma$.

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- **CA computing CA sets:**
  - enumeration model
  - a diagonalisation
  - with any simulation relation?

Still open...

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- a universal CA for onto + full-space simulations?
- a universal CA for block + context-free which is not universal when adding the 1-to-1 condition?
- a dense chain in one of the preorders?

Questions?