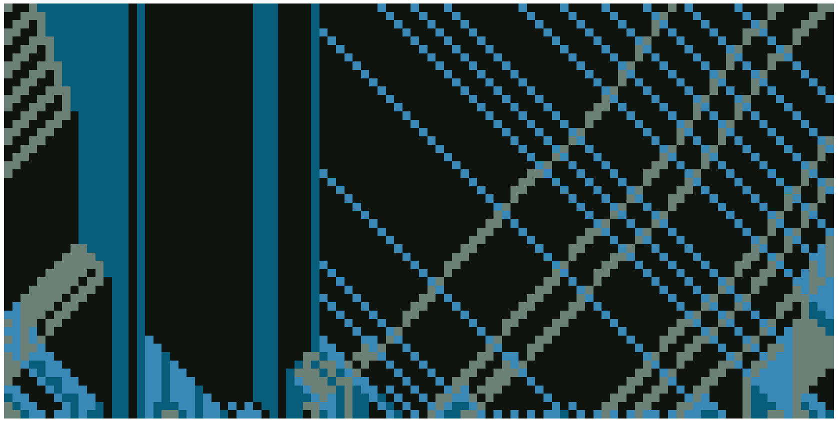
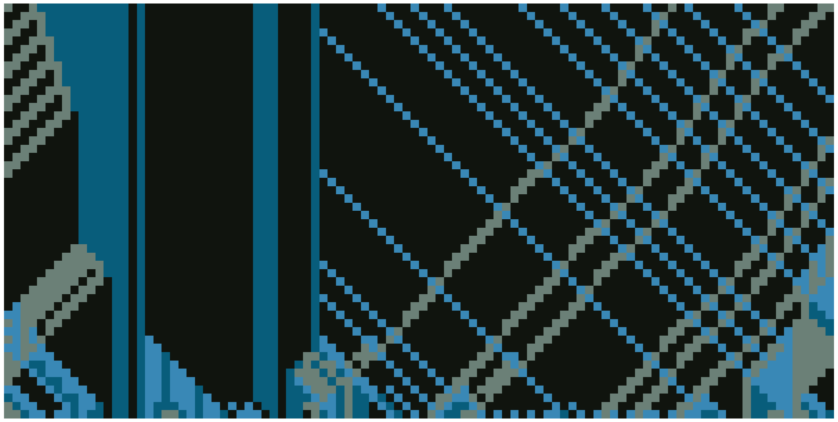
How Common Can Be Universality in Cellular Automata?



STACS 2005, Stuttgart

Guillaume Theyssier (LIP, ENS Lyon, France)

How Common Can Be **Universality** in Cellular Automata?



STACS 2005, Stuttgart

Guillaume Theyssier (LIP, ENS Lyon, France)

Smallest known universal CA:

Smallest known universal CA:

- Banks (1970): 2 states, 5 neighbours (2D)
- Ollinger (2002) : 6 states, 3 neighbours (1D)

Smallest known universal CA:

- Banks (1970): 2 states, 5 neighbours (2D)
- Ollinger (2002) : 6 states, 3 neighbours (1D)
- Cook & Wolfram : 2 states, 3 neighbours (1D)

  (only proved with a weak notion of universality)

Smallest known universal CA:

- Banks (1970): 2 states, 5 neighbours (2D)
- Ollinger (2002) : 6 states, 3 neighbours (1D)
- Cook & Wolfram : 2 states, 3 neighbours (1D)

  (only proved with a weak notion of universality)

$\rightsquigarrow$ shows how "minimal" sufficient conditions for the universality property can be, but...

Smallest known universal CA:

- Banks (1970): 2 states, 5 neighbours (2D)
- Ollinger (2002) : 6 states, 3 neighbours (1D)
- Cook & Wolfram : 2 states, 3 neighbours (1D)

  (only proved with a weak notion of universality)

$\leadsto$ shows how "minimal" sufficient conditions for the universality property can be, but...

How common is that property among CA in general?

How common is it in natural subclasses of CA?

A cellular automaton $\mathcal{A}$ is a 4-uple $(\mathbb{Z}^d, N, S, \delta)$:

$\mathbb{Z}^d$	lattice of cells
S	a finite set of states
$N = (\overrightarrow{n_1}, \ldots, \overrightarrow{n_k})$	vectors of $\mathbb{Z}^d$, neighbourhood of $\mathcal{A}$
$\delta : S^k \to S$	local transition map

A cellular automaton $\mathcal{A}$ is a 4-uple $(\mathbb{Z}^d, N, S, \delta)$:

$\mathbb{Z}^d$	lattice of cells
S	a finite set of states
$N = (\overrightarrow{n_1}, \ldots, \overrightarrow{n_k})$	vectors of $\mathbb{Z}^d$, neighbourhood of $\mathcal{A}$
$\delta : S^k \to S$	local transition map

- Configurations are mappings from $\mathbb{Z}^d$ to S.

A cellular automaton $\mathcal{A}$ is a 4-uple $(\mathbb{Z}^d, N, S, \delta)$:

$\mathbb{Z}^d$	lattice of cells
S	a finite set of states
$N = (\overrightarrow{n_1}, \ldots, \overrightarrow{n_k})$	vectors of $\mathbb{Z}^d$, neighbourhood of $\mathcal{A}$
$\delta : S^k \to S$	local transition map

- Configurations are mappings from $\mathbb{Z}^d$ to S.
- A global mapping on configurations is obtained by uniform and synchronous application of δ:

A cellular automaton $\mathcal{A}$ is a 4-uple $(\mathbb{Z}^d, N, S, \delta)$:

$\mathbb{Z}^d$	lattice of cells
S	a finite set of states
$N = (\overrightarrow{n_1}, \ldots, \overrightarrow{n_k})$	vectors of $\mathbb{Z}^d$, neighbourhood of $\mathcal{A}$
$\delta : S^k \to S$	local transition map

- Configurations are mappings from $\mathbb{Z}^d$ to S.
- A global mapping on configurations is obtained by uniform and synchronous application of δ:

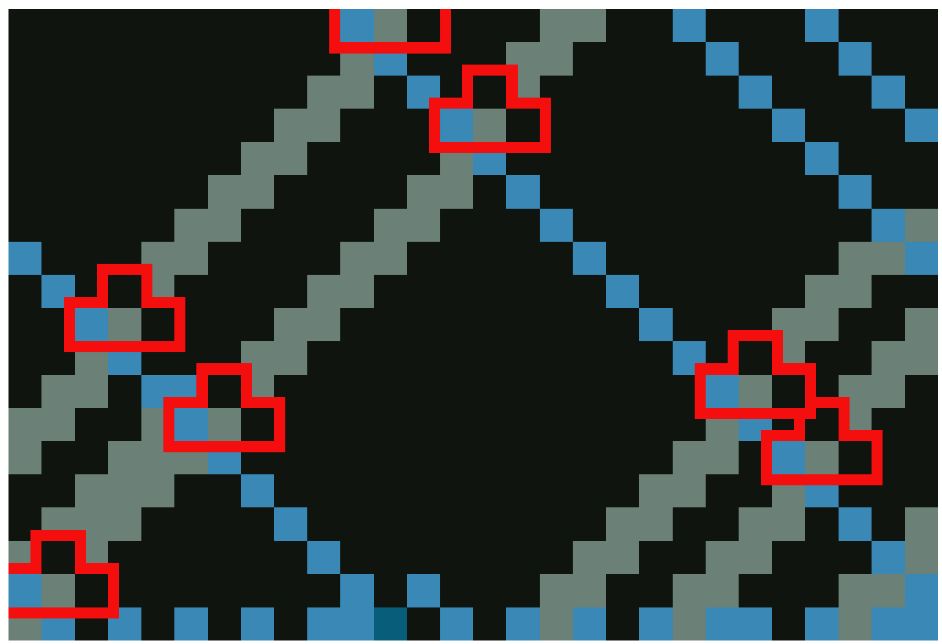
$$\forall c \in S^{\mathbb{Z}^d}, \forall \overrightarrow{z} \in \mathbb{Z}^d : \mathcal{A}(c)_{\overrightarrow{z}} = \delta(c_{\overrightarrow{z}+\overrightarrow{n_1}}, \ldots, c_{\overrightarrow{z}+\overrightarrow{n_k}})$$

Space-time diagram

$$\mathcal{A} = (\mathbb{Z}, N = \{-1, O, 1\}, S = \{\blacksquare, \blacksquare, \blacksquare, \blacksquare\}, \delta)$$

$$\delta(\blacksquare, \blacksquare, \blacksquare) = \blacksquare$$

$$\vdots$$

. . .  . . .

(time goes from bottom to top)

"how common"

- a neighbourhood is fixed
- CA_n = CA with states set $\{1, \ldots, n\}$
- $\mathrm{CA} = \bigcup \mathrm{CA}_n$

"how common"

- a neighbourhood is fixed
- CA_n = CA with states set $\{1, \ldots, n\}$
- $\mathsf{CA} = \bigcup \mathsf{CA}_n$
- A property $\mathcal{P}$ is a subset of CA

"how common"

- a neighbourhood is fixed
- CA_n = CA with states set $\{1,\ldots,n\}$
- $\mathsf{CA} = \bigcup \mathsf{CA}_n$
- A property $\mathcal{P}$ is a subset of CA
- the "probability" (or density) of $\mathcal{P}$ is

$$\mu(\mathcal{P}) = \lim_{n\to\infty} \frac{|\mathcal{P} \cap \mathsf{CA}_n|}{|\mathsf{CA}_n|} \text{ (when the limit exists)}$$

"how common"

- a neighbourhood is fixed
- CA_n = CA with states set $\{1, \ldots, n\}$
- $\mathsf{CA} = \bigcup \mathsf{CA}_n$
- A property $\mathcal{P}$ is a subset of CA
- the "probability" (or density) of $\mathcal{P}$ is

$$\mu(\mathcal{P}) = \lim_{n \to \infty} \frac{|\mathcal{P} \cap \mathsf{CA}_n|}{|\mathsf{CA}_n|} \text{ (when the limit exists)}$$

General question: what is $\mu(\mathcal{P})$ for given $\mathcal{P}$?

An example...

- the state e is quiescent for $\mathcal{A}$ if $\mathcal{A}(e, \ldots, e) = e$

An example...

- the state e is quiescent for $\mathcal{A}$ if $\mathcal{A}(e, \ldots, e) = e$
- let $\mathcal{P}$ = "having a quiescent state"

An example...

- the state e is quiescent for $\mathcal{A}$ if $\mathcal{A}(e,\ldots,e) = e$
- let $\mathcal{P}$ = "having a quiescent state"
- we have $\frac{|\mathcal{P} \cap \mathsf{CA}_n|}{\mathsf{CA}_n} = 1 - (1 - \frac{1}{n})^n$

An example...

- the state e is quiescent for $\mathcal{A}$ if $\mathcal{A}(e,\ldots,e) = e$
- let $\mathcal{P}$ = "having a quiescent state"
- we have $\frac{|\mathcal{P} \cap \mathsf{CA}_n|}{\mathsf{CA}_n} = 1 - (1 - \frac{1}{n})^n$
- thus $\mu(\mathcal{P}) = 1 - \frac{1}{e}$

A notion of stable sub-system:

$\mathcal{B}$ is a sub-automaton of $\mathcal{A}$ ($\mathcal{B} \sqsubseteq \mathcal{A}$) if

A notion of stable sub-system:

$\mathcal{B}$ is a sub-automaton of $\mathcal{A}$ ($\mathcal{B} \sqsubseteq \mathcal{A}$) if

$$\begin{array}{ccc} B^{\mathbb{Z}^d} & \xrightarrow{\bar{\iota}} & A^{\mathbb{Z}^d} \\ \downarrow \mathcal{B} & & \downarrow \mathcal{A} \\ B^{\mathbb{Z}^d} & \xrightarrow{\bar{\iota}} & A^{\mathbb{Z}^d} \end{array}$$

for some injective map $\iota : B \to A$

A notion of stable sub-system:

$\mathcal{B}$ is a sub-automaton of $\mathcal{A}$ ($\mathcal{B} \sqsubseteq \mathcal{A}$) if

$$\begin{array}{ccc} B^{\mathbb{Z}^d} & \xrightarrow{\bar{\iota}} & A^{\mathbb{Z}^d} \\ \downarrow \mathcal{B} & & \downarrow \mathcal{A} \\ B^{\mathbb{Z}^d} & \xrightarrow{\bar{\iota}} & A^{\mathbb{Z}^d} \end{array}$$

for some injective map $\iota : B \to A$

Up to renaming, $\mathcal{B}$ is $\mathcal{A}$ restricted to a subset of states

A notion of stable sub-system:

$\mathcal{B}$ is a sub-automaton of $\mathcal{A}$ ($\mathcal{B} \sqsubseteq \mathcal{A}$) if

$$\begin{array}{ccc} B^{\mathbb{Z}^d} & \xrightarrow{\bar{\iota}} & A^{\mathbb{Z}^d} \\ \downarrow \mathcal{B} & & \downarrow \mathcal{A} \\ B^{\mathbb{Z}^d} & \xrightarrow{\bar{\iota}} & A^{\mathbb{Z}^d} \end{array}$$

for some injective map $\iota : B \to A$

Up to renaming, $\mathcal{B}$ is $\mathcal{A}$ restricted to a subset of states

$\rightsquigarrow$ notion of increasing/decreasing property w.r.t. $\sqsubseteq$

“universality”

"universality"

- here universality means intrinsic universality $\neq$ Turing universality (but "intrinsic $\Rightarrow$ Turing") "universality"

- here universality means intrinsic universality $\neq$ Turing universality (but "intrinsic $\Rightarrow$ Turing")
- intrinsic universality = ability to directly simulate any CA

“universality”

- here universality means intrinsic universality $\neq$ Turing universality (but “intrinsic $\Rightarrow$ Turing”)
- intrinsic universality = ability to directly simulate any CA

$\rightsquigarrow$ need a formal definition of simulation (Rapaport-Mazoyer 98, Ollinger 02):

"universality"

- here universality means intrinsic universality $\neq$ Turing universality (but "intrinsic $\Rightarrow$ Turing")
- intrinsic universality = ability to directly simulate any CA

$\rightsquigarrow$ need a formal definition of simulation (Rapaport-Mazoyer 98, Ollinger 02):

- the $\sqsubseteq$ relation...

"universality"

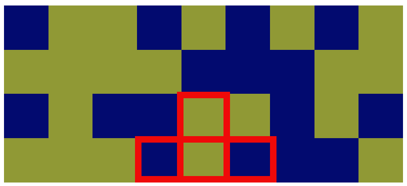
- here universality means intrinsic universality $\neq$ Turing universality (but "intrinsic $\Rightarrow$ Turing")
- intrinsic universality = ability to directly simulate any CA

$\rightsquigarrow$ need a formal definition of simulation (Rapaport-Mazoyer 98, Ollinger 02):

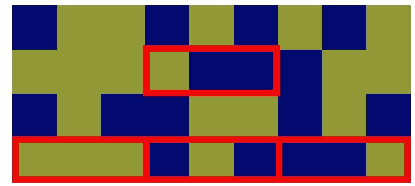
- the $\sqsubseteq$ relation...
- ...up to rescaling transformations

Rescaling transformations ($\mathcal{A} \rightarrow \mathcal{A}^{\overrightarrow{T}}$):

states: A



states: A^m



$$\mathcal{A} \xrightarrow{\overrightarrow{T}=<m,n>} \mathcal{A}^{\overrightarrow{T}}$$

$$\mathcal{A}^{<m,n>} = o^m \circ \mathcal{A}^n \circ o^{-m}$$

Rescaling transformations ($\mathcal{A} \rightarrow \mathcal{A}^{\overrightarrow{T}}$):

states: A states: A^m

$$\mathcal{A} \xrightarrow{\overrightarrow{T}=<m,n>} \mathcal{A}^{\overrightarrow{T}}$$

$$\mathcal{A}^{<m,n>} = o^m \circ \mathcal{A}^n \circ o^{-m}$$

Simulation = quasi-order $\preceq$

$$\mathcal{A} \preceq \mathcal{B} \Leftrightarrow \exists \overrightarrow{T}, \overrightarrow{T'} : \mathcal{A}^{\overrightarrow{T}} \sqsubseteq \mathcal{B}^{\overrightarrow{T'}}$$

- $\preceq$ induces an equivalence relation $\sim$

- $\preceq$ induces an equivalence relation $\sim$
- $\preceq$ admits a global maximum $\sim$-class: U

- $\preceq$ induces an equivalence relation $\sim$
- $\preceq$ admits a global maximum $\sim$-class: U
- the set $\{(\mathcal{A}, \mathcal{B}) : \mathcal{A} \preceq \mathcal{B}\}$ is recursively enumerable (but not recursive)

- $\preceq$ induces an equivalence relation $\sim$
- $\preceq$ admits a global maximum $\sim$-class: U
- the set $\{(\mathcal{A},\mathcal{B}) : \mathcal{A} \preceq \mathcal{B}\}$ is recursively enumerable (but not recursive)

Definition. $\mathcal{P}_U = \{\mathcal{A} : \forall \mathcal{B} \exists \overrightarrow{T}, \mathcal{B} \sqsubseteq \mathcal{A}^{\overrightarrow{T}}\}$ (intrinsic universality)

- $\preceq$ induces an equivalence relation $\sim$
- $\preceq$ admits a global maximum $\sim$-class: U
- the set $\{(\mathcal{A},\mathcal{B}) : \mathcal{A} \preceq \mathcal{B}\}$ is recursively enumerable (but not recursive)

Definition. $\mathcal{P}_U = \{\mathcal{A} : \forall \mathcal{B} \exists \overrightarrow{T}, \mathcal{B} \sqsubseteq \mathcal{A}^{\overrightarrow{T}}\}$ (intrinsic universality)

Remarks: $\mathcal{P}_U$ is an increasing property and $U = \mathcal{P}_U$

- $\preceq$ induces an equivalence relation $\sim$
- $\preceq$ admits a global maximum $\sim$-class: U
- the set $\{(\mathcal{A},\mathcal{B}) : \mathcal{A} \preceq \mathcal{B}\}$ is recursively enumerable (but not recursive)

Definition. $\mathcal{P}_U = \{\mathcal{A} : \forall\mathcal{B}\exists\overrightarrow{T}, \mathcal{B} \sqsubseteq \mathcal{A}^{\overrightarrow{T}}\}$ (intrinsic universality)

Remarks: $\mathcal{P}_U$ is an increasing property and $U = \mathcal{P}_U$

Question. $\mu(\mathcal{P}_U)$?

Restricting the problem to a natural subset of CA...

Restricting the problem to a natural subset of CA...

Definition. $\mathcal{A}$ is a captive cellular automaton (CCA) if every subset X of the states set induces a sub-automaton $\mathcal{A}_X$

Restricting the problem to a natural subset of CA...

Definition. $\mathcal{A}$ is a captive cellular automaton (CCA) if every subset X of the states set induces a sub-automaton $\mathcal{A}_X$

- a property of the local transition map
- a CA with 2 states is captive if and only if its 2 states are quiescent

Restricting the problem to a natural subset of CA...

Definition. $\mathcal{A}$ is a captive cellular automaton (CCA) if every subset X of the states set induces a sub-automaton $\mathcal{A}_X$

- a property of the local transition map
- a CA with 2 states is captive if and only if its 2 states are quiescent

μ' = probability restricted to CCA:

$$\mu'(\mathcal{P}) = \lim_{n \to \infty} \frac{|\mathcal{P} \cap \mathsf{CCA}_n|}{|\mathsf{CCA}_n|}$$

Theorem. For any non-trivial $\mathcal{P}$:

- $\mathcal{P}$ increasing $\Rightarrow \mu'(\mathcal{P}) = 1$
- $\mathcal{P}$ decreasing $\Rightarrow \mu'(\mathcal{P}) = 0$

Theorem. For any non-trivial $\mathcal{P}$:

- $\mathcal{P}$ increasing $\Rightarrow \mu'(\mathcal{P}) = 1$
- $\mathcal{P}$ decreasing $\Rightarrow \mu'(\mathcal{P}) = 0$

Idea of the proof: for any X of size k, choosing $\mathcal{A}$ uniformly among CCA$_n$ implies choosing $\mathcal{A}_X$ uniformly among CCA$_k$.

Theorem. For any non-trivial $\mathcal{P}$:

- $\mathcal{P}$ increasing $\Rightarrow \mu'(\mathcal{P}) = 1$
- $\mathcal{P}$ decreasing $\Rightarrow \mu'(\mathcal{P}) = 0$

Idea of the proof: for any X of size k, choosing $\mathcal{A}$ uniformly among CCA$_n$ implies choosing $\mathcal{A}_X$ uniformly among CCA$_k$. So a random $\mathcal{A} \in$ CCA$_n$ possesses n/k sub-automata chosen independently among CCA$_k$.

Theorem. For any non-trivial $\mathcal{P}$:

- $\mathcal{P}$ increasing $\Rightarrow \mu'(\mathcal{P}) = 1$
- $\mathcal{P}$ decreasing $\Rightarrow \mu'(\mathcal{P}) = 0$

Idea of the proof: for any X of size k, choosing $\mathcal{A}$ uniformly among CCA$_n$ implies choosing $\mathcal{A}_X$ uniformly among CCA$_k$. So a random $\mathcal{A} \in$ CCA$_n$ possesses n/k sub-automata chosen independently among CCA$_k$. When $n \to \infty$, the probability to avoid some fixed $\mathcal{B} \in$ CCA$_k$ is 0

Theorem. For any non-trivial $\mathcal{P}$:

- $\mathcal{P}$ increasing $\Rightarrow \mu'(\mathcal{P}) = 1$
- $\mathcal{P}$ decreasing $\Rightarrow \mu'(\mathcal{P}) = 0$

Idea of the proof: for any X of size k, choosing $\mathcal{A}$ uniformly among CCA_n implies choosing $\mathcal{A}_X$ uniformly among CCA_k. So a random $\mathcal{A} \in \text{CCA}_n$ possesses n/k sub-automata chosen independently among CCA_k. When $n \to \infty$, the probability to avoid some fixed $\mathcal{B} \in \text{CCA}_k$ is 0

Corollary 1: almost no CCA is surjective

Theorem. For any non-trivial $\mathcal{P}$:

- $\mathcal{P}$ increasing $\Rightarrow \mu'(\mathcal{P}) = 1$
- $\mathcal{P}$ decreasing $\Rightarrow \mu'(\mathcal{P}) = 0$

Idea of the proof: for any X of size k, choosing $\mathcal{A}$ uniformly among CCA$_n$ implies choosing $\mathcal{A}_X$ uniformly among CCA$_k$. So a random $\mathcal{A} \in$ CCA$_n$ possesses n/k sub-automata chosen independently among CCA$_k$. When $n \to \infty$, the probability to avoid some fixed $\mathcal{B} \in$ CCA$_k$ is 0

Corollary 1: almost no CCA is surjective

Corollary 2: $\mu'(\mathcal{P}_U) = 1$

$\mathcal{P}_U$ is almost always true... but undecidable!

$\mathcal{P}_U$ is almost always true... but undecidable!

Theorem. if the (fixed) neighbourhood is large enough, it is undecidable to know whether a given CCA is intrinsically universal or not.

$\mathcal{P}_U$ is almost always true... but undecidable!

Theorem. if the (fixed) neighbourhood is large enough, it is undecidable to know whether a given CCA is intrinsically universal or not.

Proof sketch:

- a recursive transformation τ mapping CA to CCA
- $\mathcal{A} \preceq \tau(\mathcal{A})$
- $\mathcal{A}$ and $\tau(\mathcal{A})$ simulate the same non-trivial surjective CA
- a recursive transformation χ such that
  - $\chi(\mathcal{A})$ is universal iff it can simulate all non-trivial surjective CA
  - $\chi(\mathcal{A}) \in \mathcal{P}_U$ is undecidable (Ollinger, 03)

No limit on how complex a CCA can be without being universal

No limit on how complex a CCA can be without being universal

Theorem. For any CCA $\mathcal{A} \notin \mathcal{P}_U$, there is a CCA $\mathcal{B} \notin \mathcal{P}_U$ such that $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{B} \not\preceq \mathcal{A}$

No limit on how complex a CCA can be without being universal

Theorem. For any CCA $\mathcal{A} \notin \mathcal{P}_U$, there is a CCA $\mathcal{B} \notin \mathcal{P}_U$ such that $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{B} \not\preceq \mathcal{A}$

- the semi-decidability of $\preceq$ and the undecidability of $\mathcal{P}_U$ show that there is no global maximum for CCA $\setminus \mathcal{P}_U$
- for any pair $\mathcal{A}$, $\mathcal{B}$ not in $\mathcal{P}_U$ we can construct $\mathcal{C} \notin \mathcal{P}_U$ with $\mathcal{A} \preceq \mathcal{C}$ and $\mathcal{B} \preceq \mathcal{C}$
- same result as in the general case (Ollinger, 03)

Back to the general case...

Back to the general case...

no 0-1 law (quiescent state property)

Back to the general case...

no 0-1 law (quiescent state property)

Definition. $\mathcal{P}_{\sqsubseteq} = \{\mathcal{A} : \exists \mathcal{B}, 1 < |\mathcal{B}| < |\mathcal{A}| \text{ and } \mathcal{B} \sqsubseteq \mathcal{A}\}$

Having a non-trivial sub-automaton

Back to the general case...

no 0-1 law (quiescent state property)

Definition. $\mathcal{P}_{\sqsubseteq} = \{\mathcal{A} : \exists \mathcal{B}, 1 < |\mathcal{B}| < |\mathcal{A}| \text{ and } \mathcal{B} \sqsubseteq \mathcal{A}\}$

Having a non-trivial sub-automaton

Property. $\mu(\mathcal{P}_{\sqsubseteq}) = 0$

Proof: counting argument

Back to the general case...

no 0-1 law (quiescent state property)

Definition. $\mathcal{P}_{\sqsubseteq} = \{\mathcal{A} : \exists \mathcal{B}, 1 < |\mathcal{B}| < |\mathcal{A}| \text{ and } \mathcal{B} \sqsubseteq \mathcal{A}\}$

Having a non-trivial sub-automaton

Property. $\mu(\mathcal{P}_{\sqsubseteq}) = 0$

Proof: counting argument

- a “random” CA has no local structure
- arguments for CCA do not apply in general

Open questions:

(i). is there some $\mathcal{A}$ such that $\forall \overrightarrow{T}$, $\mathcal{A}^{<\overrightarrow{T}>} \notin \mathcal{P}_{\sqsubseteq}$?

(ii). $\mu(\mathcal{P}_U)$?

- $\mu(\mathcal{P}_U) = c > 0$
- not defined
- $\mu(\mathcal{P}_U) = 0$ (and possibly almost all CA at the bottom for $\preceq$?)

(iii). what are $\sim$-classes containing some CCA?