Directional Dynamics of Cellular Automata

M. Delacourt, V. Poupet, M. Sablik and *G. Theyssier*

LIF / LIF / LATP / LAMA
(CNRS, Université de Provence)$\times$3, (CNRS, Université de Savoie)

September 1st 2010

Overview of the talk

1. Cellular Automata
2. Topological Dynamics
3. Directional Dynamics

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1. **Cellular Automata**
2. Topological Dynamics
3. Directional Dynamics

Cellular automata

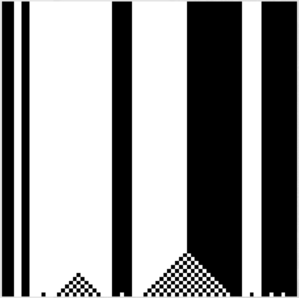
Definition

- **Syntactical object:**
  - Q state set,
  - $V = \{v_1, \ldots, v_k\} \subseteq \mathbb{Z}$ neighborhood,
  - $f : Q^V \to Q$ local transition rule
- **Associated behavior:**
  - $Q^{\mathbb{Z}}$: set of configurations
  - $F : Q^{\mathbb{Z}} \to Q^{\mathbb{Z}}$ global transition rule defined by

$$F(x)_z = f\left(x_{z+v_1}, \ldots, x_{z+v_k}\right)$$

Cellular automata

Examples



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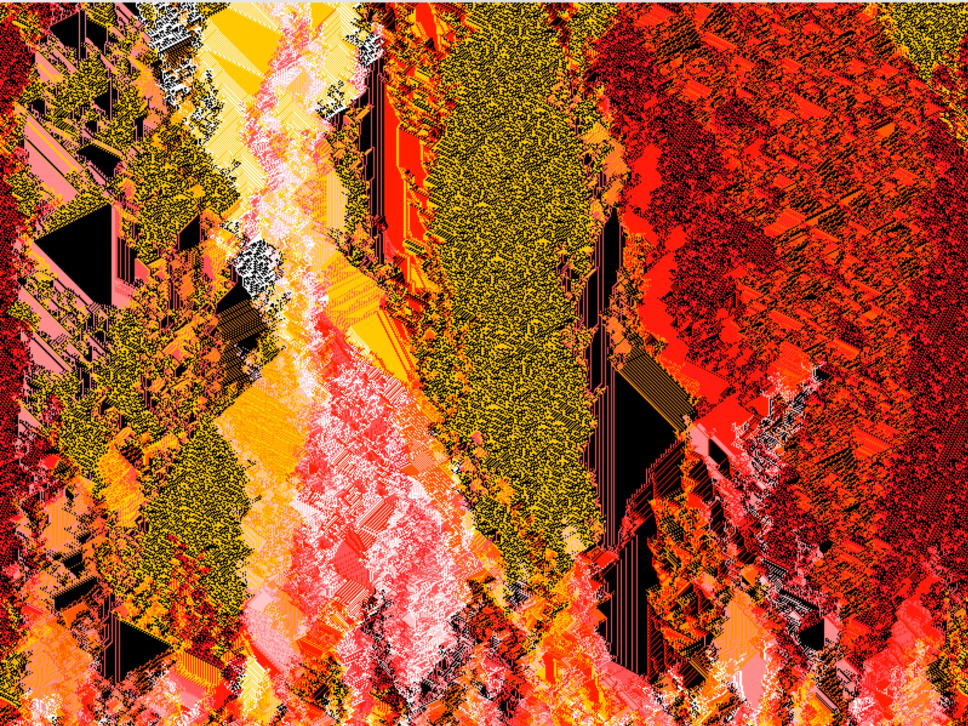

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1 Cellular Automata

2 Topological Dynamics

3 Directional Dynamics

CA as Dynamical Systems

- **Cantor distance:**
  - $D(x, y)$= dist. to center of the 1st cell where x and y differ
  - Cantor distance: $d(x, y) = 2^{-D(x,y)}$
  - the space of configurations is compact

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Theorem (Curtis, Hedlund, Lyndon, 69)

CA global functions are exactly the continuous function which commute with shift maps.

- shift map of vector z: $\sigma_z(x) = z' \mapsto x_{z+z'}$
- **intuition**: continuity $\equiv$ short-term predictability

Deterministic Chaos

- δ: precision on initial conditions
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Sensitivity to initial conditions:

$$\exists\epsilon, \forall x, \forall\delta, \exists y, \exists n : d(x, y) \leq \delta \text{ and } d(F^n(x), F^n(y)) \geq \epsilon$$

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Equicontinuity point at x:

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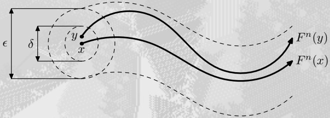

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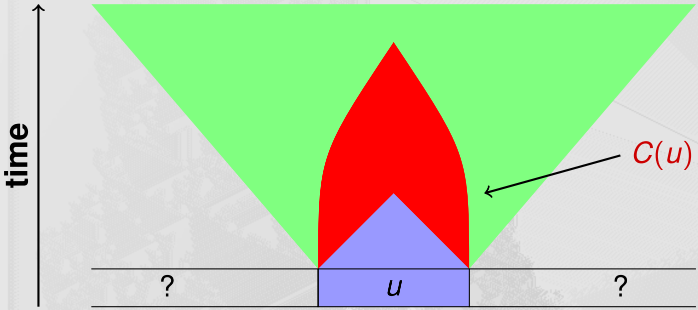
- **Eq** $\stackrel{def}{=}$ set of CA having equicontinuity points

Equicontinuity without topology

Consequences $C(u)$ of a word u



$$[u] = \{x \in Q^{\mathbb{Z}} : x_0 \cdots x_{|u|-1} = u\}$$

$$C(u) = \{(z,t) : \forall x, y \in [u], F^t(x)_z = F^t(y)_z\}$$

Equicontinuity without topology

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- F a 1D CA with radius r
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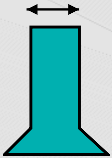

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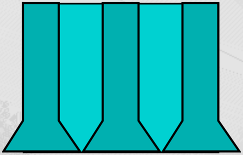

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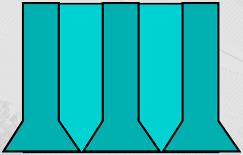
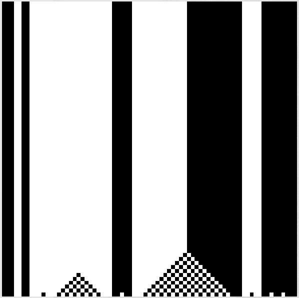


Theorem (P. Kůrka, 1997)

*$F \in$ **Eq** $\iff$ F has a wall $\iff$ F is almost everywhere equicontinuous*

Examples: do they have walls?



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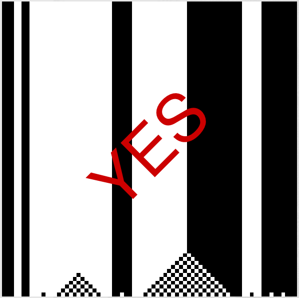

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Undecidability

The Usual Refrain

Theorem (Durand-Formenti-Varouchas, 2003)

It is undecidable to know whether a given CA has a wall or not.

(I've been told a short proof of this using Kari's undecidability result on nilpotency)

Corollary

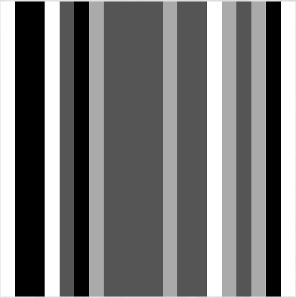
There is no computable bound on the size of the smallest wall.

Open problem

Is the set of 1D CA having a wall a Σ_2-complete set?

Problem with Shift Invariance

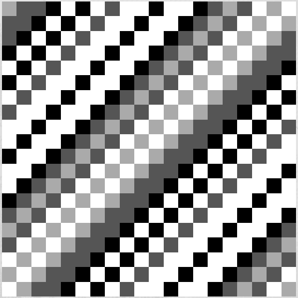
Identity



equicontinuous

long-term predictable

Shift



sensitive

NOT long-term predictable?

Topological dynamics with Cantor metrics not well adapted!

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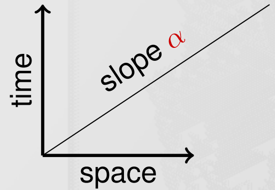
The Work of M. Sablik (2008)



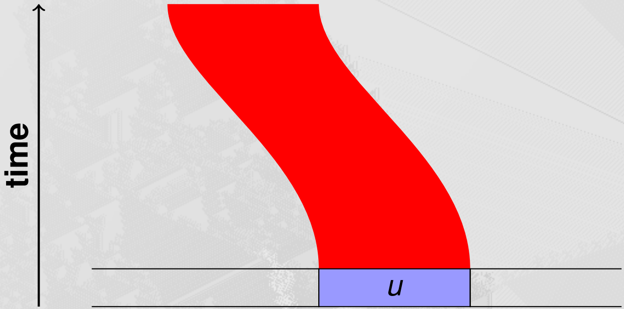
1. for any α, consider the 'sloped' system: $t \mapsto \sigma_{\lceil \alpha t \rceil} \circ F^t$
2. for each fixed α, Kůrka's results hold
3. study the set of slopes α showing a given behavior

Example of Theorem

For any given CA, the set of slopes with equicontinuity points is an interval of reals.

Generalization

- Walls along arbitrary curves



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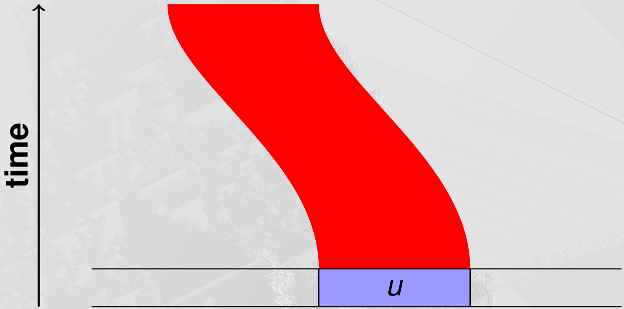

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- **useful definition? do 'non-linear walls' exist?**

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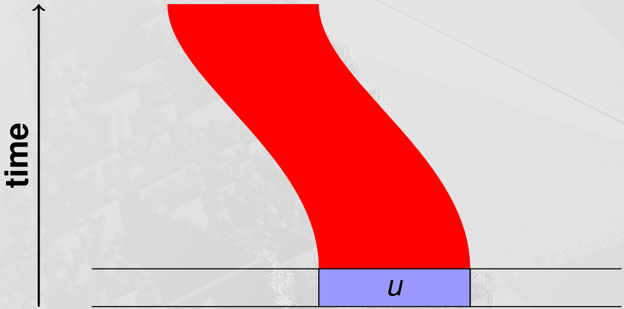
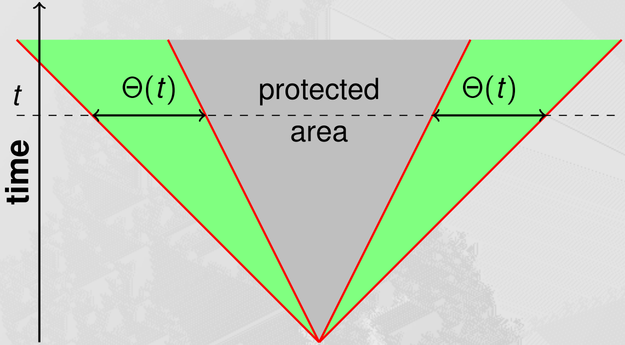
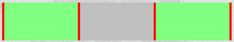


- theory still works (wall $\leftrightarrow$ equicontinuity) but...
- **useful definition? do 'non-linear walls' exist?**
- this talk: YES + other strange behaviors

The Key Construction

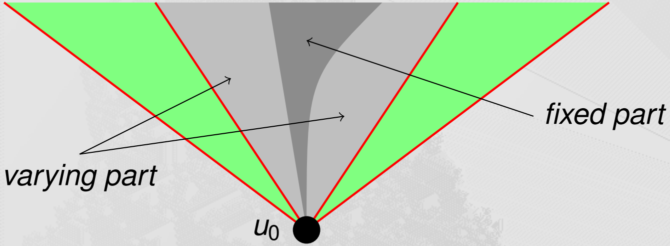


Valid zone $\stackrel{def}{=}$ 

1. only a valid zone can erase a valid zone
2. after t steps, the age of any valid zone is at least t
3. when two valid zones meet, the older is destroyed

The Key Construction

Evil Twins



- $u_0 \stackrel{def}{=}$ protected zone of age 0

Principle

If the construction can merge with itself **then** $C(u_0)$ is exactly the fixed part of the construction.

The Key Construction

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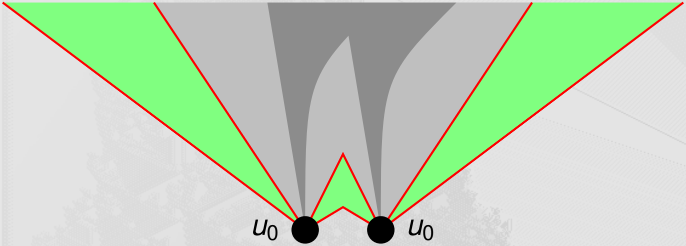

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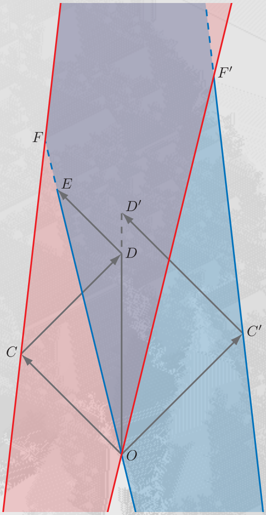
Details



Result 1

Parabolic Consequences

Theorem

There exists a CA having a wall along a parabolic curve, but without any wall along any linear direction.

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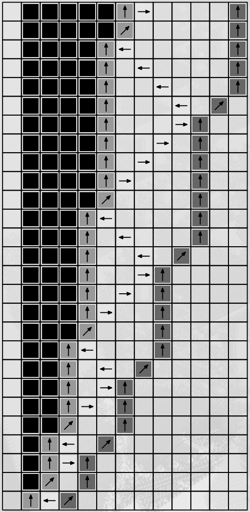
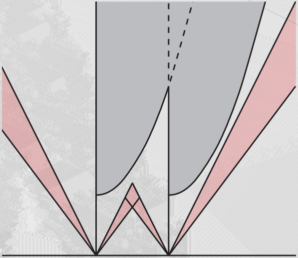
Main ideas

- two constructions
- intersection via Cartesian product
- protected zones

Result 1

2 pictures about the proof





Result 2

Characterization of linear slopes

Definition

A real α is computably enumerable (c.e.) if there is a computable sequence of rationals converging to it.

- left c.e. if the sequence is increasing
- right c.e. if the sequence is decreasing

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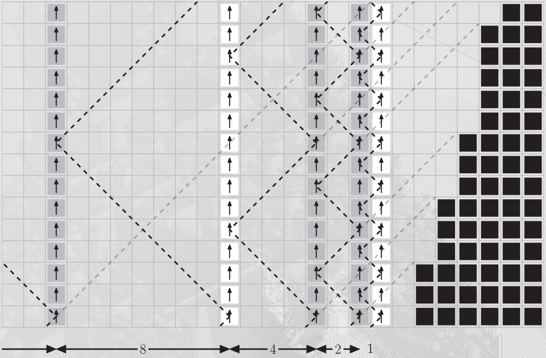
Theorem

$[\alpha, \beta]$ is the set of slope of equicontinuity for some CA **iff** α is lce and β is rce.

Result 2

1 picture about the proof

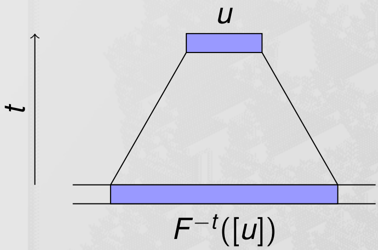


$$\alpha = \sum_i \frac{\alpha_i}{2^i}$$

(here: $\alpha_i = 0$ iff $i = 0$ or $i = 2$)

Result 3

Strange Typical Behaviors



- μ: Bernouilli measure
- $\mu_t(u) \stackrel{def}{=} \mu\left(F^{-t}([u])\right)$
- $\chi = t \mapsto t \bmod 2$

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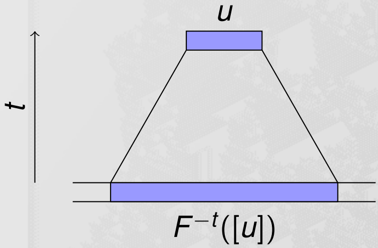

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Proposition

There exist a CA F with:

1. $\mu_t(0) \to_t \chi$
2. $\mu_t(1) \to_t 1 - \chi$
3. $\mu_t(q) \to_t 0$ for any $q \notin \{0, 1\}$

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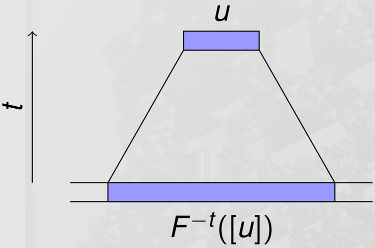

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- u **persistent** $\stackrel{def}{\iff} \mu_t(u) \not\to 0$

Work in progress (with Sablik *et al.*)

Any transitive regular language can be the persistent language of some CA.

Other results + many details

Directional Dynamics along Arbitrary Curves in Cellular Automata
Delacourt, Poupet, Sablik, Theyssier
37p. To appear in TCS.

Future Work

- directional expansivity
- restriction to surjective CA
- what sets of configurations can be a μ-limit-set?
- what sets of configurations cannot be a μ-limit-set?