Captive Cellular Automata

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Abstract. We introduce a natural class of cellular automata characterised by a property of the local transition law without any assumption on the states set. We investigate some algebraic properties of the class and show that it contains intrinsically universal cellular automata. In addition we show that Rice's theorem for limit sets is no longer true for that class, although infinitely many properties of limit sets are still undecidable.

Cellular automata (CA for short) are discrete dynamical systems capable of producing a wide class of different behaviours. They consist of a large collection of simple identical components (the cells) with uniform local interactions. As such they provide an idealistic model to study complex systems observed in nature. Despite the simplicity of the model, most of the richness of behaviours they exhibit is still to be understood. Moreover, many interesting and natural properties are undecidable. To that extent it is meaningful to consider classes of CA obtained by structural assumptions on the local transition law with the hope that these local assumptions are sufficiently handleable to express global dynamical properties of CA from this class.

To our knowledge, the main attempt in that sense appearing in literature is the class of additive CA first suggested by O. Martin, A. M. Odlyzko and S. Wolfram in [1]. Additive CA are those which are linear with respect to some commutative ring structure on the state set. Thanks to classical algebraic tools, such CA are now well understood on several aspects. Unfortunately, they don't reveal the richness of CA both dynamically and algorithmically (in particular, they cannot be intrinsically universal). Notice that other classes, like threshold CA studied in [2], were defined using an interpretation of the states as weights and are now well-understood.

In this paper, we define a new class of CA, namely *captive cellular automata* (CCA for short), which relies on a characterisation of their local transition law without endowing the state set with any external structure. This characterisation is based on a natural notion of sub-dynamical system in the context of CA, the notion of sub-automaton, which has often been considered and, particularly, play a central role in the algebraic classifications of CA introduced by J. Mazoyer and I. Rapaport in [3] and generalised by N. Ollinger in [4]. CCA are CA

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following a canonical form with respect to that notion: precisely those for which the transition law is stable for any set of states. By means of this strong characterisation, we investigate properties of Cca which show that this class makes sense on several aspects.

First we prove that it possesses interesting closure properties (especially reversibility) and we fully characterise Cca which are also additive or permutive. Then, we construct for any ca a Cca capable of simulating it, what shows that there exists intrinsically universal Cca. We also show that Cca are non-trivially distributed with respect to the algebraic classification mentioned above. Finally, we consider Rice's theorem for limit sets (established by J. Kari in [5]) and show that it does not hold for the class of Cca, although an unavoidable set of infinitely many properties are still undecidable due to the ability of Cca to uniformly simulate ca.

1 Definitions and Notations

Formally, a cellular automaton $\mathcal{A}$ of dimension d (d-ca) is given by a state set A, a tuple of neighbourhood vectors $(\overrightarrow{v}_1, \ldots, \overrightarrow{v}_n)$ and a local transition map f from A^n to A. A cell is indexed by a position $\overrightarrow{z}$ of the lattice $\mathbb{Z}^d$. A configuration of $\mathcal{A}$ is a mapping from the lattice $\mathbb{Z}^d$ to the state set A giving a state to each cell. When the dimension is fixed, $A^{\mathbb{Z}^d}$ will be called the set of A-configurations. On each discrete time step, the cells alter their states synchronously according to f. This defines a mapping $\mathcal{A}$ from configurations to configurations as follows:

$$\forall c \in A^{\mathbb{Z}^d}, \forall \overrightarrow{z} \in \mathbb{Z}^d : \big(\mathcal{A}(c)\big)(\overrightarrow{z}) = f\big(c(\overrightarrow{z} + \overrightarrow{v}_1), \ldots, c(\overrightarrow{z} + \overrightarrow{v}_n)\big).$$

Global maps on configurations which are actual global maps of ca have been characterised topologically by M. L. Curtis, G. A. Hedlund and R. C. Lyndon in [6] (they give the characterisation for dimension 1, and D. Richardson extends the result to any dimension in [7]). The set of configurations endowed with the natural topology is a compact space. Then the global maps of ca are precisely the continuous maps which commute with the translations (for any $\overrightarrow{t} \in \mathbb{Z}^d$, the translation $\sigma_{\overrightarrow{t}}$ is defined by $\sigma_{\overrightarrow{t}}(c)(\overrightarrow{i} + \overrightarrow{t}) = c(\overrightarrow{i})$). Notice that this characterisation justifies the fact that the inverse map of a ca global map is itself a ca global map.

Without loss of generality we consider only ca whose neighbourhood is a ball, for the infinite norm on $\mathbb{Z}^d$, centred on $(0, \ldots, 0)$. When considering a ca $\mathcal{A}$, the notation $\mathcal{A}$ denotes at the same time the local transition map and the global map on configurations, A denotes the state set and $r_{\mathcal{A}}$ the radius of the ball neighbourhood of $\mathcal{A}$.

The limit set $\Omega_{\mathcal{A}}$ of a ca $\mathcal{A}$ is the set of configurations that may appear at any time step in the evolution of $\mathcal{A}$ (for a more complete study of limit sets, see [8]). Formally,

$$\Omega_{\mathcal{A}} = \bigcap_{t \in \mathbb{N}} \mathcal{A}^t\big(A^{\mathbb{Z}^d}\big).$$

It is always non-empty but it can be a singleton. It has also the important property that for any configuration $c \in \Omega_{\mathcal{A}}$, there is $d \in \Omega_{\mathcal{A}}$ such that $\mathcal{A}(d) = c$. When $\Omega_{\mathcal{A}}$ is reduced to a single configuration, $\mathcal{A}$ is said to be *nilpotent*. The nilpotency problem is undecidable in any dimension (the first proof for dimension $d \geq 2$ appears in [8], later a proof for any dimension was given in [9]).

Let $\mathcal{A}$ be a CA and $A' \subseteq A$. The fact that A' is stable under the action of $\mathcal{A}$ (namely $\mathcal{A}((A')^{\mathbb{Z}^d}) \subseteq (A')^{\mathbb{Z}^d}$) is denoted by $A' \sqsubseteq \mathcal{A}$. The restriction of $\mathcal{A}$ to A' is then called a *sub-automaton* of $\mathcal{A}$. More generally, a CA $\mathcal{B}$ is a *sub-automaton* of $\mathcal{A}$ (also denoted by $\mathcal{B} \sqsubseteq \mathcal{A}$) if there is some $A' \sqsubseteq \mathcal{A}$ such that $\mathcal{B}$ is isomorphic to the restriction of $\mathcal{A}$ to A'.

Definition 1. *$\mathcal{A}$ is a* captive cellular automaton *(*CCA *for short) if it satisfies the following equivalent conditions:*

1. $\forall B \subseteq A : B \sqsubseteq \mathcal{A}$;
2. $\forall B \subseteq A : |B| = |A| - 1 \Rightarrow B \sqsubseteq \mathcal{A}$;
3. $\forall a_1, \ldots, a_{(2r_{\mathcal{A}}+1)^d} \in A : \mathcal{A}(a_1, \ldots, a_{(2r_{\mathcal{A}}+1)^d}) \in \{a_1, \ldots, a_{(2r_{\mathcal{A}}+1)^d}\}$.

Those equivalent conditions will often be referred to as "the jail property". It is clear that this is a locally checkable property. Notice that the identity map and more generally any shift map is trivially a CCA. Besides, a 2-state CA is a CCA if and only if each of its states is quiescent (q is quiescent if the uniform configuration $\overline{q}$ is a fixed point).

To end this section, we assume from this point on and without explicit mention that any CA considered will be one-dimensional and with at least 2 states. To simplify notations when considering a CA $\mathcal{A}$ of radius r, $\mathcal{A}$ will denote not only the local and the global map but also the action of the CA over the set of words over the alphabet with length at least $2r_{\mathcal{A}} + 1$. A word of length $2r_{\mathcal{A}} + 1$ will be called a *neighbourhood word* for $\mathcal{A}$. We denote by $L(c)$ the set of words appearing in the configuration c. Finally, for any word w let $\Sigma(w)$ denote the set of letters appearing in it.

2 First Properties

We will now present some simple properties implied by the strong structure of the local transition map of a CCA.

First it is straightforward to verify that CCA are closed under composition and sub-automata (any sub-automaton of a CCA is itself a CCA) but not by Cartesian product. An interesting and less immediate closure property is that of inversion. We give here a one-dimensional proof for clarity in notations but it is straightforward to extend it to any dimension.

Proposition 1. *Let $\mathcal{A}$ be a* CCA *which is reversible and denote by $\mathcal{A}^{-1}$ its inverse* CA. *Then $\mathcal{A}^{-1}$ is a* CCA.

Proof. Let r be the radius of $\mathcal{A}^{-1}$ and consider any $(2r+1)$-tuple of states $(x_{-r}, \ldots, x_r)$. Let c be a periodic configuration of period $x_{-r} \ldots x_r$. The sequence of configurations $\left(\mathcal{A}^n(c)\right)_{n\in\mathbb{N}}$ is periodic (the sequence cannot be only ultimately periodic because $\mathcal{A}$ is bijective). Then there is $n \in \mathbb{N}$ such that $\mathcal{A}^n(c) = c$. Therefore $\mathcal{A}^{-1}(x_{-r} \dots x_r) \in \{x_{-r}, \dots, x_r\}$ because $\mathcal{A}^{n-1}(c) = \mathcal{A}^{-1}(c)$ and $\mathcal{A}^{n-1}$ has the jail property. □

We now give two illustrations of other constraints induced by the jail property. We characterise CCA which are respectively additive and permutive. First, for the additive case, we consider the following definition which clearly contains the linear CA of [1].

Proposition 2. *Let $\mathcal{A}$ be a* CCA. *If we can endow the states set A with a group law denoted by $+$ with neutral element 0 such that*

$$\forall c, c' \in A^{\mathbb{Z}},\ \mathcal{A}(c\overline{+}c') = \mathcal{A}(c)\overline{+}\mathcal{A}(c')$$

(where $\overline{+}$ is the uniform extension of $+$ to configurations), then either $\mathcal{A}$ is a shift map or A has only 2 states.

Proof. Let $k = 2r_{\mathcal{A}} + 1$ and for $1 \le i \le k$ denote by $f_i : A \to A$ the map

$$x \mapsto \mathcal{A}(\underbrace{0, \dots, 0}_{i-1}, x, \underbrace{0, \dots, 0}_{k-i}).$$

By hypothesis, $\forall x_1, \dots, x_k \in A$, $\mathcal{A}(x_1, \dots, x_k) = \sum_{i=1}^{k} f_i(x_i)$. Moreover, $\mathcal{A}$ being a CCA, $\forall x$ and $\forall i$, either $f_i(x) = 0$ or $f_i(x) = x$. Let i be fixed and consider x such that $f_i(x) = x$ and y such that $f_i(y) = 0$. Then

$$f_i(x + y) = 0 \Rightarrow x = 0 \text{ and } f_i(x + y) = x + y \Rightarrow y = 0.$$

So either $f_i = Id$ or $f_i = 0$. Let now suppose that the group A has an element a such that $a + a \ne a$ and $a + a \ne 0$. Then there is at most one i such that $f_i = Id$. Indeed, if $f_i = f_j = Id$ with $i \ne j$, $\mathcal{A}$ cannot satisfy the jail property on the input $x_1, \dots, x_k$ with $x_i = x_j = a$ and $x_m = 0$ for $m \ne i$, $m \ne j$. In this case, either $\mathcal{A}$ is a shift, or the constant map equal to 0, but a CCA with a least 2 states cannot be constant.

Otherwise, each element $a \ne 0$ of A is of order 2. Then if $\mathcal{A}$ is neither a shift nor the constant map equal to 0 (*i.e.* if there is $i \ne j$ such that $f_i = f_j = Id$), A cannot contain 2 distinct elements a and b different from 0 because $\mathcal{A}$ would not satisfy the jail property on the entry $x_1, \dots, x_k$ with $x_i = a$, $x_j = b$ and $x_m = 0$ for $m \ne i$, $m \ne j$: $a + b$ can neither be 0 (otherwise $a = -b = b$), nor a (otherwise $b = 0$), nor b (otherwise $a = 0$). Thus A has only 2 elements. □

The notion of permutivity was essentially studied through LR-permutive CA which are simple examples of expansive CA (see [10] for example). A proposition similar to proposition 2 for permutivity can be stated as follows.

Proposition 3. *A* CA $\mathcal{A}$ *is said to be* permutive at position i *($1 \le i \le 2r + 1$) if for all $u \in A^{i-1}$ and all $v \in A^{2r+1-i}$ the map $\pi_{i,u,v} : x \mapsto \mathcal{A}(uxv)$ is bijective from A to A. If $\mathcal{A}$ is a* CCA *which is permutive at 2 positions $i \ne j$, then $\mathcal{A}$ has only 2 states.*

Proof. Without loss of generality one can suppose $i < j$. Consider $u \in A^{i-1}$, $v \in A^{2r+1-i}$ and $P = \Sigma(uv)$. Let I denote the set $\pi_{i,u,v}^{-1}(P)$. The map $\pi_{i,u,v}^{-1}$ is bijective by hypothesis so $|P| = |I|$. Consider $x \in I \setminus P$. We have $x \notin P$ and $\pi_{i,u,v}^{-1}(x) \neq x$ (because $\pi_{i,u,v}(x) \in P$), then $x = \mathcal{A}\big(u\pi_{i,u,v}^{-1}(x)v\big) \notin \Sigma\big(u\pi_{i,u,v}^{-1}(x)v\big)$ which contradicts the jail property. Thus $I \setminus P = \emptyset$ and hence $I = P$. Therefore for any $a \notin P$, $\mathcal{A}(uav) \notin P$ because $a \notin I$, so $\mathcal{A}(uav) = a$ by the jail property. The same is true at position j.

Suppose now a, b, c are 3 different elements in A; according to what was shown above $a = \mathcal{A}(b^{i-1}ab^{j-i-1}cb^{2r+1-j}) = c$: contradiction. So $|A| = 2$. □

3 Simulations and Universality

From previous section, Cca appear to be very constrained and could constitute a marginal class of ca (as additive ca). In this section we will show that it is not the case: Cca actually exhibit as much richness and complexity as ca in general. To give a precise sense to the latter assertion, we rely on the algebraic framework introduced by J. Mazoyer and I. Rapaport ([3]) and extended by N. Ollinger ([11]). It consists in a quasi-order over ca interpreted as a simulation relation: a ca can simulate another if the second is a sub-automaton of the first, up to "rescaling" of both. It naturally induces an equivalence relation and an order on equivalence classes which provides an interesting formal tool to compare ca. The generalised notion of rescaling introduced by N. Ollinger leads to an order with a global maximum which correspond to the set of intrinsically universal ca, that is ca which can simulate any ca. Notice that this notion captures intuitive ideas already present in the work of E. R. Banks (see [12]) and formalised for the first time by J. Albert and K. Čulik in [13]. We refer the reader to [4] (in French) or [14] for a deeper study of such ideas.

The notion of rescaling can be uniformly formalised in any dimension but for clarity we give only the definition for one-dimensional ca. For any ca $\mathcal{A}$ and any positive integer m denote by o^m the "packing" map from A-configurations to A^m-configurations:

$$\forall c \in A^{\mathbb{Z}}, \forall z \in \mathbb{Z} : \big(o^m(c)\big)(z) = \big(c(mz), c(mz+1), \ldots, c(m(z+1)-1)\big).$$

For any ca $\mathcal{A}$ and any $k \in \mathbb{Z}$, $m > 0$ and $n > 0$ let $\mathcal{A}^{<m,n,k>}$ denote the (m,n,k)-rescaling of $\mathcal{A}$ defined as follows: $\mathcal{A}^{<m,n,k>} = \sigma^k \circ o^m \circ \mathcal{A}^n \circ (o^m)^{-1}$. Then $\mathcal{A}$ can simulate $\mathcal{B}$ (denoted by $\mathcal{A} \preceq \mathcal{B}$) if there exist $k, k' \in \mathbb{Z}$ and $m, m', n, n' > 0$ such that $\mathcal{B}^{<m,n,k>} \sqsubseteq \mathcal{A}^{<m',n',k'>}$. Let us now show how any ca can be simulated in that precise sense by an appropriate Cca.

Definition 2. *Let $\mathcal{A}$ be a* ca *of radius r and state set $A = \{a_1, \ldots, a_n\}$ and let $\#$ be an additional state not in A. Let $w = w_1 w_2 \ldots w_{n+2}$ denote the word $\#a_1 a_2 \ldots a_n\#$. Then $\kappa_{\#,A}$ denote the following injective coding map from $A^{\mathbb{Z}}$ to $(A \cup \{\#\})^{\mathbb{Z}}$:*

$$\big(\kappa_{\#,A}(c)\big)(z) = \begin{cases} c(k) & \text{if } z = k(n+3) \text{ for } k \in \mathbb{Z}, \\ w_i & \text{otherwise where } i = z \bmod (n+3). \end{cases}$$

We define $\rho_\#(\mathcal{A})$, a CA *of radius $r(n+3)+n+2$ and state set $A \cup \{\#\}$. It acts as follows on any neighbourhood word $u = u_{-r(n+3)-n-2} \cdots u_{r(n+3)+n+2}$:*

$$\big(\rho_\#(\mathcal{A})\big)(u) = \begin{cases} \mathcal{A}(c_{-r} \dots c_0 \dots c_r) & \text{if } u = wc_{-r}w \dots wc_iw \dots wc_rw, \\ \# & \text{if } u \notin L\big(\kappa_{\#,A}(A^{\mathbb{Z}})\big) \text{ and } \# \in \Sigma(u) \\ u_0 & \text{otherwise.} \end{cases}$$

Informally, $\kappa_{\#,A}(c)$ consist in the bi-infinite concatenation of macro-cells of the form $m_c((i)) = c(i)\#a_1 \cdots a_n\#$, where $m_c(i)$ begins at position $i(n+3)$ in $\kappa_{\#,A}(c)$ and encodes the cell at position i in the configuration c.

A word for $\rho_\#(\mathcal{A})$ is said to be *valid* if it is a sub-word of $\kappa_{\#,A}(c)$ for some $c \in A^{\mathbb{Z}}$. In such a word, a letter is said to be *informative* if it is the first letter of a macro-cell. A valid neighbourhood word is *active* if its central cell is informative. Here is an illustration of those notions for $n = 2$ and $r = 1$:

$$\underbrace{\#a_1a_2\#a_1\#a_1a_2\#a_2\#a_1a_2\#a_2\#a_1a_2\#}_{\text{active}}a_1\#a_1a_2\#a_1\#a_1a_2\#a_2\#a_1a_2\,\underbrace{\#a_2\#a_2}_{\text{non-valid}}$$

(valid but not active: $a_1a_2\#a_2\#a_1a_2\#a_1\#a_1a_2\#a_1\#a_1a_2\#a_2$)

Then $\rho_\#(\mathcal{A})$ acts as follows on a neighbourhood word u:

- if u is active then $\rho_\#(\mathcal{A})$ mimics $\mathcal{A}$ using informative letters present in u;
- if u is not valid and contains # then the cell state becomes #;
- no change is done in other cases.

It is thus clear that $\rho_\#(\mathcal{A})$ is a CCA: updates of the cell state are done in such a way that the new cell state was already present in the neighbourhood word. The construction gives raise to simulation of any CA by a CCA (and it can be straightforwardly extend to higher dimensions).

Proposition 4. *For any* CA *$\mathcal{A}$, we have $\mathcal{A} \preceq \rho_\#(\mathcal{A})$.*

Proof. We actually show that $\mathcal{A} \sqsubseteq \rho_\#(\mathcal{A})^{<n+3,1,0>}$. By construction, we have the following commutative diagram:

$$\begin{array}{ccccc} A^{\mathbb{Z}} & \xrightarrow{\kappa_{\#,A}} & \kappa_{\#,A}\big(A^{\mathbb{Z}}\big) & \xrightarrow{o^{n+3}} & M^{\mathbb{Z}} \\ \big\downarrow \mathcal{A} & & \big\downarrow \rho_\#(\mathcal{A}) & & \big\downarrow \rho_\#(\mathcal{A})^{<n+3,1,0>} \\ A^{\mathbb{Z}} & \xrightarrow{\kappa_{\#,A}} & \kappa_{\#,A}\big(A^{\mathbb{Z}}\big) & \xrightarrow{o^{n+3}} & M^{\mathbb{Z}} \end{array}$$

where M denotes the set of all macro-cells in the sense of the coding map $\kappa_{\#,A}$. The construction guaranties that $M \sqsubseteq \rho_\#(\mathcal{A})^{<n+3,1,0>}$. □

We deduce from proposition 4 that CCA reach the top of the order on CA, that is there exists intrinsically universal CCA. We can also show that the order, restricted to the classes containing CCA, is still rich. More precisely, it contains infinite chains and admits any finite tree as a sub-order. These two facts are straightforward corollaries of the following proposition.

Proposition 5. *There exists a family* $\left(\mathcal{B}_{m,n}\right)_{m,n\in\mathbb{N},n\geq m}$ *of* CCA *such that:*

- $(m \leq m') \wedge (n \leq n) \Rightarrow \mathcal{B}_{m,n} \preceq \mathcal{B}_{m',n'}$;
- $(n < n') \vee (m < m') \Rightarrow \mathcal{B}_{m',n'} \not\preceq \mathcal{B}_{m,n}$

Proof. For each m, n let $A_{m,n} = \{1, \ldots, m+n\}$ and $\mathcal{A}_{m,n}$ be a CCA of radius 5 on the state set $A_{m,n} \cup \{\#\}$ defined hereafter. Let $w = w_1w_2w_3w_4$ be the word $\#m\#\#$. Among configurations of $\mathcal{A}_{m,n}$ we distinguishes the following set of configurations: $S^0_{m,n} = \{c \in (A_{m,n} \cup \{\#\})^{\mathbb{Z}} : c(i) = w_{i \bmod 5}$ if $i \neq 0 \bmod 5\}$. Let $S_{m,n} = \bigcup_{i\in\mathbb{Z}} \sigma^i(S^0_{m,n})$. Then $\mathcal{A}_{m,n}$ is defined by

$$\mathcal{A}_{m,n}(u_{-5}\ldots u_5) = \begin{cases} \max(x,y,z) & \text{if } u = xwywz \text{ and } \max(x,y,z) \geq m \\ x & \text{if } u = xwxwx \text{ and } x \leq m \\ m & \text{otherwise if } u = xwywz \text{ and } x,y,z \leq m \\ u_0 & \text{otherwise if } u \in L(S_{m,n}) \\ \# & \text{if } u \notin L(S_{m,n}) \text{ and if } \# \in u \\ \min(u_i) & \text{in any other case.} \end{cases}$$

Now, for any CA $\mathcal{A}$, let $Y(\mathcal{A})$ be the set of cycles in the phase space of $\mathcal{A}$, and let $U(\mathcal{A})$ be the set of cycles containning only configurations with a unique predecessor.

We claim that $\left|Y(\mathcal{A}^{<a,b,c>}_{m,n})\right| = 6(m+n)+1$ and $\left|U(\mathcal{A}^{<a,b,c>}_{m,n})\right| = 5(m-1)$ for any rescaling parameters a, b and c. To complete the proof from that claim it is sufficient to notice that:

1. $m \leq m'$ and $n \leq n' \Rightarrow \mathcal{A}_{m,n} \sqsubseteq \mathcal{A}_{m',n'}$;
2. for any CA $\mathcal{A}$ and $\mathcal{B}$: $\mathcal{A} \sqsubseteq \mathcal{B} \Rightarrow \begin{cases} |Y(\mathcal{A})| \leq |Y(\mathcal{B})| \text{ and} \\ |Y(\mathcal{A}) \setminus U(\mathcal{A})| \leq |Y(\mathcal{B}) \setminus U(\mathcal{B})| \end{cases}$.

Then define $\mathcal{B}_{m,n} = \mathcal{A}_{m,n-m}$.

To prove the claim, first notice that rescaling a CA $\mathcal{A}$ does not change the cardinalities of the sets Y and U. Thus the proof is brought down to enumerating $Y(\mathcal{A}_{m,n})$ and $U(\mathcal{A}_{m,n})$. This is a straightforward case study from the definition of $\mathcal{A}_{m,n}$. The reader can verify that configurations from the sets $S_{m,n}$, $A^{\mathbb{Z}}_{m,n}$ and $X = \{c : c \notin S_{m,n} \cup A^{\mathbb{Z}}_{m,n}\}$ have disjoint orbits. Thus, considering $\mathcal{A}_{m,n}$ on each of these sets separately, we can easily infer :

- $Y(\mathcal{A}_{m,n}) \cap S_{m,n} = \{c : c$ is periodic of period xw with $x \in A_{m,n}\}$;
- $U(\mathcal{A}_{m,n}) \cap S_{m,n} = \{c : c$ is periodic of period xw with $x \in A_{m,n}, x < m\}$;
- $Y(\mathcal{A}_{m,n}) \cap A^{\mathbb{Z}}_{m,n} = \{\overline{x} : x \in A_{m,n}\}$;
- $Y(\mathcal{A}_{m,n}) \cap X = \{\overline{\#}\}$;
- $U(\mathcal{A}_{m,n}) \cap A^{\mathbb{Z}}_{m,n} = U(\mathcal{A}_{m,n}) \cap X = \emptyset$.

To conclude the proof, just notice that for any $x \in A_{m,n}$ there are exactly 5 periodic configurations of period xw. □

4 Decidability

We will now present some decidability results. The situation concerning CCA is somewhat balanced because on one hand many undecidable properties for CA are preserved under the coding schemes introduced in section 3, and on the other hand the nilpotency problem used in many proofs by reduction becomes trivial in the context of CCA (a CCA $\mathcal{A}$ has at least 2 configurations in its limit set because for any state s the uniform configuration $\overline{s}$ is a fixed point.) We now give precise decidability results.

We will focus on an analog of Rice's theorem for CA limit sets established by J. Kari in [5]. It states that, in any dimension, any non-trivial property of limit sets is undecidable. More precisely J. Kari shows that the nilpotency problem can be reduced to any decision problem concerning a non-trivial property of limit sets. As said before, there are no nilpotent CCA. Thus it is natural to ask what Kari's theorem becomes when restricted to CCA. We are going to show that there is as many undecidable properties for CCA as undecidable properties for CA in general. This result actually relies on the properties of the maps $\kappa_{\#,A}$ and $\rho_{\#}$, while the jail property is not directly used in the proof. So we present it in the general framework of any set of CA. Before stating the theorem let us formalise the notion of *property of limit sets*.

To avoid irrelevant set theoretical problems (due to renaming of states), we will temporally assume like in [5] that any CA state set comes from a countably infinite set of states $S = \{s_0, s_1, \ldots\}$. For any finite set $T \subseteq S$ let P_T denote the set of all subsets of $T^{\mathbb{Z}}$ (P_T contains all possible limit sets involving only states from T). Finally let U be the infinite union of sets P_T for all finite T. A property $\mathcal{P}$ of limit sets is a subset of U and a CA $\mathcal{A}$ has the property $\mathcal{P}$ (denoted by $\mathcal{A} \vdash \mathcal{P}$) if $\Omega_{\mathcal{A}} \in \mathcal{P}$.

Theorem 1. *Let E be a set of* CA *such that there are maps ρ and $\kappa_{\mathcal{A}}$ with the following properties:*

- *ρ is a computable map from the set of* CA *to E ;*
- *for any* CA *$\mathcal{A}$, $\kappa_{\mathcal{A}}$ maps the configurations of $\mathcal{A}$ to configurations of $\rho(\mathcal{A})$ and the following diagram commutes:*

$$\begin{array}{ccc} A^{\mathbb{Z}} & \xrightarrow{\kappa_{\mathcal{A}}} & \kappa_{\mathcal{A}}\big(A^{\mathbb{Z}}\big) \\ \Big\downarrow \mathcal{A} & & \Big\downarrow \rho(\mathcal{A}) \\ A^{\mathbb{Z}} & \xrightarrow{\kappa_{\mathcal{A}}} & \kappa_{\mathcal{A}}\big(A^{\mathbb{Z}}\big) \end{array}$$

- *κ maps are overall injective: $\kappa_{\mathcal{A}}(c) = \kappa_{\mathcal{B}}(d) \Rightarrow \kappa_{\mathcal{A}} = \kappa_{\mathcal{B}}$ and $c = d$;*
- *κ maps are honest: if $\rho(\mathcal{A})\big(d\big) = c$ then $c \in \kappa_{\mathcal{B}}(B^{\mathbb{Z}}) \Rightarrow d \in \kappa_{\mathcal{A}}(A^{\mathbb{Z}})$.*

Then there is an injective map Φ on U such that, for any non-trivial property $\mathcal{P}$, $\Phi(\mathcal{P})$ is an undecidable property when restricted to the set E of input CA.

Proof. The map Φ is defined as follows:

$$\Phi(\mathcal{P}) = \bigcup_{\mathcal{A}} \big\{P_+ \in U : P_+ \cap \kappa_{\mathcal{A}}(A^{\mathbb{Z}}) \in \{\kappa_{\mathcal{A}}(P) : P \in \mathcal{P}, P \neq \emptyset\}\big\}.$$

First notice that Φ is injective because if $\mathcal{P}_a \neq \mathcal{P}_b$ in U, there is $P_a \in \mathcal{P}_a$, $P_a \notin \mathcal{P}_b$ (the proof is the same if $\mathcal{P}_b \setminus \mathcal{P}_a \neq \emptyset$). Let $\mathcal{A}$ be a CA such that $P_a \subseteq A^{\mathbb{Z}}$ and let $K = \kappa_{\mathcal{A}}(P_a)$. Clearly $K \in \Phi(\mathcal{P}_a)$. Suppose $K \in \Phi(\mathcal{P}_b)$. Then there are $P \in \mathcal{P}_b$, $P \neq \emptyset$ and a CA $\mathcal{B}$ such that $K \cap \kappa_{\mathcal{B}}(B^{\mathbb{Z}}) = \kappa_{\mathcal{B}}(P)$. By overall injectivity of κ maps, this implies $\mathcal{A} = \mathcal{B}$. Therefore $\kappa_{\mathcal{A}}(P) = K$ and hence, by injectivity of $\kappa_{\mathcal{A}}$, $P_a = P \in \mathcal{P}_b$: contradiction.

Let us now show that $\mathcal{A} \vdash \mathcal{P}$ if and only if $\rho(\mathcal{A}) \vdash \Phi(\mathcal{P})$ (what is sufficient to complete the proof by Kari's theorem since ρ is computable). First suppose $\Omega_{\mathcal{A}} \in \mathcal{P}$. We have by the commutative diagram $\kappa_{\mathcal{A}}(\Omega_{\mathcal{A}}) \subseteq \Omega_{\rho(\mathcal{A})} \cap \kappa_{\mathcal{A}}(A^{\mathbb{Z}})$. Conversely for any $c \in \Omega_{\rho(\mathcal{A})} \cap \kappa_{\mathcal{A}}(A^{\mathbb{Z}})$ there is $d \in \Omega_{\rho(\mathcal{A})}$ such that $\big(\rho(\mathcal{A})\big)(d) = c$ (by the property of limit sets mentioned in section 1). Therefore by the honesty property $d \in \kappa_{\mathcal{A}}(A^{\mathbb{Z}})$. We can repeat this argument to construct an infinite history of configurations in $\kappa_{\mathcal{A}}(A^{\mathbb{Z}})$ leading to c by $\rho(\mathcal{A})$. By the commutative diagram, we deduce that $c \in \kappa_{\mathcal{A}}(\Omega_{\mathcal{A}})$. Thus $\Omega_{\rho(\mathcal{A})} \cap \kappa_{\mathcal{A}}(A^{\mathbb{Z}}) \subseteq \kappa_{\mathcal{A}}(\Omega_{\mathcal{A}})$ and hence, $\Omega_{\rho(\mathcal{A})} \cap \kappa_{\mathcal{A}}(A^{\mathbb{Z}}) = \kappa_{\mathcal{A}}(\Omega_{\mathcal{A}})$. Therefore $\Omega_{\rho(\mathcal{A})} \in \Phi(\mathcal{P})$ by definition of Φ.

Suppose now that $\Omega_{\rho(\mathcal{A})} \in \Phi(\mathcal{P})$. Then there are $P \in \mathcal{P}$, $P \neq \emptyset$ and a CA $\mathcal{B}$ such that $\Omega_{\rho(\mathcal{A})} \cap \kappa_{\mathcal{B}}(B^{\mathbb{Z}}) = \kappa_{\mathcal{B}}(P)$. As above, for any $c \in \Omega_{\rho(\mathcal{A})} \cap \kappa_{\mathcal{B}}(B^{\mathbb{Z}})$, we have $d \in \Omega_{\rho(\mathcal{A})}$ such that $\big(\rho(\mathcal{A})\big)(d) = c$. By the honesty property, this implies $d \in \kappa_{\mathcal{A}}(A^{\mathbb{Z}})$ and $c \in \kappa_{\mathcal{A}}(A^{\mathbb{Z}})$. Thus, by overall injectivity, $\kappa_{\mathcal{A}} = \kappa_{\mathcal{B}}$ (and therefore $A = B$). We have shown above that $\Omega_{\rho(\mathcal{A})} \cap \kappa_{\mathcal{A}}(A^{\mathbb{Z}}) = \kappa_{\mathcal{A}}(\Omega_{\mathcal{A}})$. So we have the following:

$$\kappa_{\mathcal{A}}(\Omega_{\mathcal{A}}) = \Omega_{\rho(\mathcal{A})} \cap \kappa_{\mathcal{A}}(A^{\mathbb{Z}}) = \Omega_{\rho(\mathcal{A})} \cap \kappa_{\mathcal{B}}(B^{\mathbb{Z}}) = \kappa_{\mathcal{B}}(P) = \kappa_{\mathcal{A}}(P).$$

By injectivity of $\kappa_{\mathcal{A}}$, we deduce that $\Omega_{\mathcal{A}} = P \in \mathcal{P}$. □

Let us now show how to construct κ and ρ maps for CCA. Some technical points must be made more precise. First, the macro-cells used in the construction of $\kappa_{\#,A}$ rely on an enumeration of the state set A. Given that any CA considered here take its states from S we will always use enumerations following the order of elements in S. We will also shift the states to free s_0 which will play the role of the special state $\#$. Formally, let s denote the successor map on S ($s(s_i) = s_{i+1}$) and $\overline{s}$ its extension to configurations. ρ and κ maps are then defined as follows

$$\rho(\mathcal{A}) = \rho_{s_0}\big(\overline{s} \circ \mathcal{A} \circ \overline{s}^{-1}\big), \qquad \kappa_{\mathcal{A}} = \kappa_{s_0,\mathcal{A}} \circ \overline{s}.$$

Lemma 1. *The maps defined above fill the hypotheses of theorem 1 for the set of* CCA.

Proof. First, it is clear that ρ is computable and proposition 4 shows that the diagram in the hypotheses commutes. Let us show that κ maps are overall injective and honest. Let $\mathcal{A}$ and $\mathcal{B}$ be two CA on state sets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_l\}$ enumerated in increasing order according to S.

Suppose that $c \in A^{\mathbb{Z}}$ and $d \in B^{\mathbb{Z}}$ are such that $\kappa_{\mathcal{A}}(c) = \kappa_{\mathcal{B}}(d)$. Necessarily $k = l$ because if $k < l$ (the other case is symmetric) then $\big(\kappa_{\mathcal{A}}(c)\big)(k+2) = s_0$ and $\big(\kappa_{\mathcal{B}}(d)\big)(k+2) = b_{k+1}$, but $s_0 \neq b_{k+1}$ because $b_{k+1} > b_1 \geq s_0$. Then equality of configurations $\kappa_{\mathcal{A}}(c)$ and $\kappa_{\mathcal{B}}(d)$ at positions 2 to $k+1$ implies $A = B$ and therefore $\kappa_{\mathcal{A}} = \kappa_{\mathcal{B}}$. Finally, $c = d$ because $\kappa_{\mathcal{A}}(c)$ and $\kappa_{\mathcal{B}}(d)$ are equal at position $z(k+3)$ for any $z \in \mathbb{Z}$. This shows overall injectivity. For honesty, suppose that $c \in \kappa_{\mathcal{B}}(B^{\mathbb{Z}})$ and $d \in \big(s(A) \cup \{s_0\}\big)^{\mathbb{Z}}$ are such that $\rho(\mathcal{A})\big(d\big) = c$. Let $\mathcal{R} = \rho(\mathcal{A})$. First $\mathcal{R}$ has the following property :

$$(\mathcal{R}(x)(i-1) \neq s_0) \wedge (\mathcal{R}(x)(i) = s_0) \wedge (\mathcal{R}(x)(i+1) \neq s_0) \Rightarrow x(i) = s_0.$$

Indeed, if $x(i) \neq s_0$ and $\mathcal{R}(x)(i) = s_0$ the the neighbourhood of cell i in x is non-valid and contains a s_0. Then either cell $i-1$ or cell $i+1$ has a non-valid neighbourhood which contains s_0 (it is straightforward to verify): thus either $\mathcal{R}(x)(i-1) = s_0$ or $\mathcal{R}(x)(i+1) = s_0$. We deduce that $c(i) = s_0 \Rightarrow d(i) = s_0$ because $s_0 s_0 \notin L(c)$ (c is a valid configuration for $\kappa_{\mathcal{B}}$). In particular, any neighbourhood word for $\mathcal{R}$ in d contains s_0. Let us suppose that $d \notin \kappa_{\mathcal{A}}(A^{\mathbb{Z}})$. Then there is $z \in \mathbb{Z}$ such that position z in d is non-valid and contains s_0 in its neighbourhood. Therefore $\mathcal{R}(d)(z) = c(z) = s_0$ and, according to what was shown above, $d(z) = s_0$. Either position $z-1$ or $z+1$ is non-valid and has s_0 in its neighbourhood (precisely at position z): in both case we have exhibited an occurrence of the word $s_0 s_0$ in c which contradicts its validity. □

This lemma together with theorem 1 shows that there is an infinite number of undecidable properties of limit sets for CCA. However, the analog of Rice's theorem is no longer true for CCA. For instance, the property "an even number of different states appears in the limit set" is clearly decidable for CCA although non-trivial, simply because each state of a CCA appears in its limit set.

Let us now present a natural problem on limit sets which is undecidable for CA but decidable for CCA. We no longer follows the hypothesis of states being taken in S. Let $\mathcal{A}$ be a fixed CA of radius r.

Proposition 6. *Consider the following decision problems :*

($\mathcal{P}_1$) *Input : a* CA *$\mathcal{B}$ of radius r. Question : $\Omega_{\mathcal{A}} = \Omega_{\mathcal{B}}$?*
($\mathcal{P}_2$) *Input : a* CCA *$\mathcal{B}$ of radius r. Question : $\Omega_{\mathcal{A}} = \Omega_{\mathcal{B}}$?*

$\mathcal{P}_1$ is undecidable whereas $\mathcal{P}_2$ is decidable.

Proof. First let X denote the set of states appearing in $\Omega_{\mathcal{A}}$. Notice that a CCA $\mathcal{B}$ is such that $\Omega_{\mathcal{B}} = \Omega_{\mathcal{A}}$ only if $B = X$. Thus there is only a finite number of possible CCA having the same limit set as $\mathcal{A}$ and the problem $\mathcal{P}_2$ is trivially decidable. Now we prove that $\mathcal{P}_1$ is undecidable. We proceed by a reduction from the nilpotency problem which is proven undecidable in [9] even if we restrict ourselves to CA of radius r with a spreading state (a state q is spreading for $\mathcal{N}$ if $\mathcal{N}(u) = q$ for any neighbourhood word u containing an occurrence of q). Now given any CA $\mathcal{N}$ of radius r with a spreading state q, we can algorithmically construct a CA $\mathcal{D}$ as follows. First (up to renaming) we can suppose $A \cap N = \emptyset$.

Let $D = A \cup (N \setminus \{q\})$ be the state set of $\mathcal{D}$. Let f and g denote the following maps from D to N and form N to D (respectively), where $a_0 \in A$ is fixed:

$$f : x \mapsto \begin{cases} q & \text{if } x \in A, \\ x & \text{otherwise,} \end{cases} \qquad g : y \mapsto \begin{cases} a_0 & \text{if } y = q, \\ y & \text{otherwise.} \end{cases}$$

Then $\mathcal{D}$ is defined by

$$\mathcal{D}(x_{-r}, \ldots, x_r) = \begin{cases} \mathcal{A}(x_{-r}, \ldots, x_r) & \text{if } x_i \in A, -r \leq i \leq r, \\ g\big(\mathcal{N}(f(x_{-r}), \ldots, f(x_r))\big) & \text{otherwise.} \end{cases}$$

Then $\Omega_{\mathcal{D}} = \Omega_{\mathcal{A}}$ if and only if $\mathcal{N}$ is nilpotent. Indeed, if $\mathcal{N}$ is nilpotent there is n_0 such that for all c, $\mathcal{N}^{n_0}(c) = \overline{q}$. Therefore $\mathcal{D}^{n_0}(D^{\mathbb{Z}}) \subseteq A^{\mathbb{Z}}$ and $\Omega_{\mathcal{D}} \subseteq \Omega_{\mathcal{A}}$. So $\Omega_{\mathcal{D}} = \Omega_{\mathcal{A}}$ because $\mathcal{D}$ is equal to $\mathcal{A}$ on $A^{\mathbb{Z}}$. Conversely, if $\mathcal{N}$ is not nilpotent, then there is $c \neq \overline{q}$ in $\Omega_{\mathcal{N}}$. Then for any $t \in \mathbb{N}$ there is $c_t \in N^{\mathbb{Z}}$ such that $\mathcal{N}^t(c_t) = c$. Let $d_t = \overline{g}(c_t)$ (where $\overline{g}$ is the uniform extension of g to configurations). Clearly for any $i \in \mathbb{Z}$ such that $c(i) \neq q$, we have $\big(\mathcal{D}^t(d_t)\big)(i) \notin A$. By compactness, there is $d \in D^{\mathbb{Z}}$ with infinite history for $\mathcal{D}$ and such that $d \notin A^{\mathbb{Z}}$: thus $\Omega_{\mathcal{D}}$ cannot be equal to $\Omega_{\mathcal{A}}$. □

5 Conclusion and Perspectives

The model of CA is widely admitted as a relevant framework to study questions raised by the paradigm of complex systems. Unfortunately, most of the interesting properties concerning their behaviours are undecidable. It is then natural to consider sub-classes of CA with the hope that classical problems and generally behaviours classification will be easier when restricted to that class. CCA introduced here constitute such a class, with both the property that some undecidable problems become decidable when restricted to the class (section 4) and that it somehow preserves the complexity of CA (section 3).

We defined CCA by a formal property of the local transition law. A remarkable fact is that reversible CCA satisfy the property both forwards and backwards. Is there a topological proof of the stability by inversion of the jail property? And more generally is there a topological characterisation of CCA?

Besides, we emphasise the non-closure of CCA by Cartesian product and a noticeable consequence: the classical simulation of any CA by a one-way CA cannot be directly transfered to CCA. More generally, the role played by the radius and the number of states are highly non-symmetric in the context of CCA, contrary to the general case. Is there a hierarchy of complexity for CCA according to the radius?

We also showed that undecidability remains highly present in CCA. The frontier between decidability and undecidability is thus still to be made more precise. Particularly, are classical properties like that of reaching the limit set in finite time or being reversible in dimension 2 still undecidable for CCA?

Finally, experiments[1] on randomly chosen Cca show particular dynamics (for example, equicontinuity points appears very often) and propositions 2 and 3 suggests that “chaotic-like” behaviours are extremely constrained for Cca. Can we give a more precise sense to this assertion? For instance, can we explicit the intersections of Cca with Kůrka's classes (introduced in [15])?

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[1] See `http://captive.ca.free.fr` for random space-time diagrams of Cca