# Amalgamation of Cellular Automata

## JAC 2008 — *Médoc* session

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# Motivations

- **The real one:** how common is (intrinsic) universality in CA?

- **Others:**
  1. amalgamations as composition operations
     - state efficient algorithmic constructions
     - alternative to cartesian product
  2. sub-famillies of CA obtained by syntactical constructions
     - study "malleability" of universality
     - study density of various properties
     - a hope to gain on decidability

# Setting of the talk

- dimension is 1 (not necessary for most results)
- **a neighbourhood is fixed** (Von Neumann radius $r$)
- state set may vary but is always of the form $\{1, \ldots, n\}$

# Preliminary definitions

- **Stability**
  - given $F$ over state set $Q$
  - $Q_0 \subset Q$ is $F$-stable if $F\left(Q_0^{\mathbb{Z}}\right) \subseteq Q_0^{\mathbb{Z}}$
  - the induced automaton is denoted by $F_{|Q_0}$

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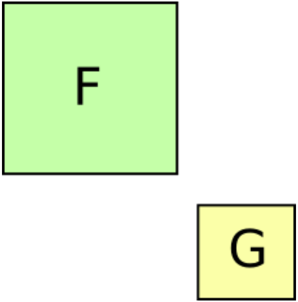
- **Sub-automaton**
  - given $F$ and $G$ over $Q_F$ and $Q_G$ (resp.)
  - **notation:** $F \sim G$ if they are equal upto state renaming
  - **definition:** $F \sqsubseteq G$ (sub-automaton) if

$$\exists Q \subseteq Q_G : Q \text{ is } G\text{-stable and } G_{|Q} \sim F$$

# Amalgamations

- consider two CA:
  - $F$ over $Q_F$
  - $G$ over $Q_G$
  - $Q_F \cap Q_G = \emptyset$



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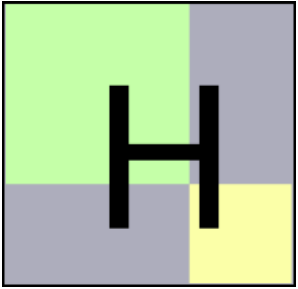


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  - $F$ over $Q_F$
  - $G$ over $Q_G$
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- $H$ is an amalgamation of $F$ and $G$ if:
  - $Q_H = Q_F \cup Q_G$
  - $Q_F$ and $Q_G$ are both $H$-stable
  - $H_{|Q_F} = F$
  - $H_{|Q_G} = G$

- **Amalgamation operation:**
  - $\Gamma : CA \times CA \to 2^{CA}$
  - $\Gamma(F, G)$ is a set of amalgamations of $F$ and $G$

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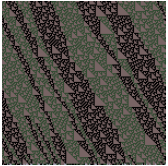
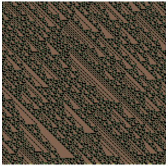
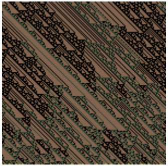
- **Amalgamation family:**
  - $\mathbb{G}$ a finite set of CA
  - $\Gamma$ an amalgamation operation
  - associated family $\mathcal{F}(\Gamma, \mathbb{G})$ is the closure of $\mathbb{G}$ by $\Gamma$

# Amalgamations

## An example: scarf CA

# Amalgamations

## An example: scarf CA

- given $F_1$ and $F_2$
- choose:
  1. $G$ with 2 states
  2. $\phi_1 : Q_{F_1} \cup Q_{F_2} \to Q_{F_1}$ (identity over $Q_{F_1}$)
  3. $\phi_2 : Q_{F_1} \cup Q_{F_2} \to Q_{F_2}$ (identity over $Q_{F_2}$)
- the **scarf** amalgamation $H$ is defined by

$$H(x_{-r}, \dots, x_r) =$$

$$\begin{cases} F_1(\phi_1(x_{-r}), \dots, \phi_1(x_r)) & \text{if } G(\chi(x_{-r}), \dots, \chi(x_r)) = 1 \\ F_2(\phi_2(x_{-r}), \dots, \phi_2(x_r)) & \text{if } G(\chi(x_{-r}), \dots, \chi(x_r)) = 2 \end{cases}$$

where $\chi(x) = i \iff x \in Q_{F_i}$.

## Asymptotic density

- $\mathcal{F}$ is a family (a set of CA with above conditions)
- $\mathcal{P}$ is a property (another set)

- Density of $\mathcal{P}$ in $\mathcal{F}$ :

$$d_{\mathcal{F}}(\mathcal{P}) = \lim_{n\to\infty} \frac{\#\mathcal{F}_n \cap \mathcal{P}}{\#\mathcal{F}_n}$$

- $\mathcal{F}_n$ is the set of CA from $\mathcal{F}$ with $n$ states
- the limit is not always defined

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**Fact**

*If $\mathcal{F}$ is an amalgamation family, then $d_{CA}(\mathcal{F}) = 0$*

# Simulation and universality

- **Simulation:**
  - **intuition:** sub-automaton relation up to space-time rescaling
  - **definition:** $F \preceq G$ iff

$$F \preceq G \iff \exists t, t', m, m' : \mathbf{b}_m \circ F^t \circ \mathbf{b}_m^{-1} \sqsubseteq \mathbf{b}_{m'} \circ G^{t'} \circ \mathbf{b}_{m'}^{-1}$$

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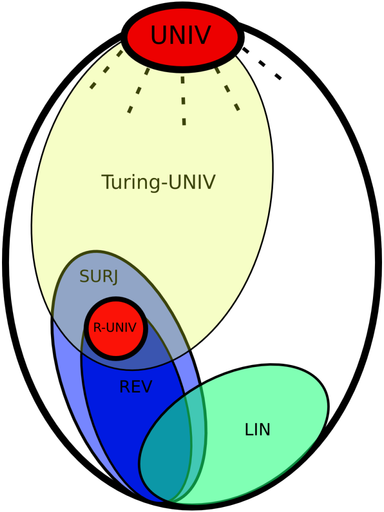
- **Intrinsic universality:**

**Definition**

*$F$ is universal if for all $G$ we have $G \preceq F$*

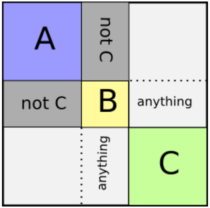
# Simulation and universality



# Amalgamations

## Unambiguous families

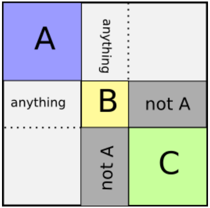
- an amalgamation operation is generally not associative




$\Gamma\big(\Gamma(A, B), C\big)$

$\not\subseteq$

$\not\supseteq$

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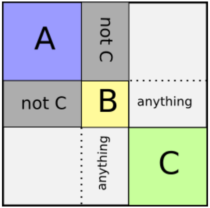


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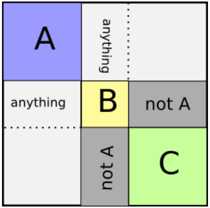


$\Gamma\big(A,\Gamma(B,C)\big)$

- unambiguous amalgamation family $\mathcal{F}(\Gamma,\mathbb{G})$:
  1. $\Gamma$ is associative
  2. $\#\Gamma(F,G)$ is a function of $|F|$ and $|G|$ only
  3. there is no $G_1, G_2 \in \mathbb{G}$ with $G_1 \sqsubseteq G_2$

# Amalgamations

## A class of properties with high density

- $\mathcal{F}(\Gamma, \mathbb{G})$ an unambiguous amalgamation family
- a property $\mathcal{P}$ is *malleable* if
  1. $\exists i, \forall G_1, \ldots, G_i \in \mathbb{G} : \Gamma(G_1, \ldots, G_i) \cap \mathcal{P} \neq \emptyset$
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**Proposition**

*If $\mathcal{F}(\Gamma, \mathbb{G})$ is unambiguous and $\mathcal{P}$ is a malleable property then $d_{\mathcal{F}}(\mathcal{P}) = 1$.*

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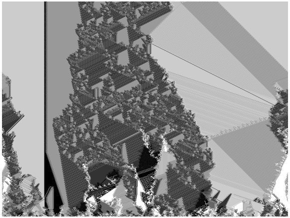
**Proposition**

*If $\mathcal{F}(\Gamma, \mathbb{G})$ is unambiguous and $\mathcal{P}$ is a malleable property then $d_{\mathcal{F}}(\mathcal{P}) = 1$.*

**Proof ingredients:**

- partition $\mathcal{F}_n$ according to generator sequence
- balance of amalgamation application trees

# Examples: Captive CA



- **local constraint**

$(x_1, \ldots, x_k) \mapsto y \in \{x_1, \ldots, x_k\}$

- $\Gamma_{\mathcal{K}}$: transition tables completed according to the constraint
- $\mathcal{K} = \mathcal{F}(\Gamma_{\mathcal{K}}, \{\mathbb{1}\})$

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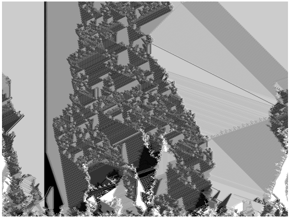

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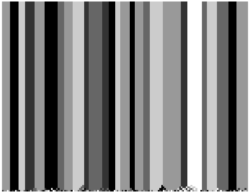
**Fact**

- *$\Gamma_{\mathcal{K}}$ is associative*
- *any $\mathcal{P}$ increasing for $\preceq$ and non-trivial in $\mathcal{K}$ is malleable for $\mathcal{F}(\Gamma_{\mathcal{K}}, \mathbb{G})$*

# Examples: Majority CA



▶ **local constraint**

$$(x_1, \ldots, x_k) \mapsto y \in \{x_i : c(i) \geq c(j), \forall j\}$$

where $c(i) = \#\{j : x_j = x_i\}$

- $\Gamma_{\mathcal{M}_{aj}}$: transition tables completed according to the constraint
- $\mathcal{M}_{aj} = \mathcal{F}(\Gamma_{\mathcal{M}_{aj}}, \{\mathbb{1}\})$

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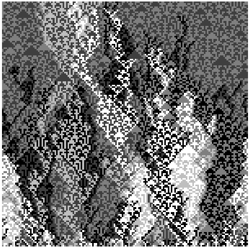

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# Encodings

## Definition

- let $\mathcal{F}_1$ and $\mathcal{F}_2$ be families of CA
- $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ is an encoding if
  1. it is computable and injective
  2. $\forall F \in \mathcal{F}_1 : F \preceq \phi(F)$
- it is *faithful* if $F \in \mathcal{U} \iff \phi(F) \in \mathcal{U}$

▶ **Remarks**

- an encoding preserves existence of universal CA
- a faithful encoding preserves undecidability of $\mathcal{U}$

## Proposition

$$
\begin{array}{ccccccc}
 & & & & \mathcal{M}_{aj} & \rightarrow & \mathcal{F}(\Gamma_{\mathcal{M}_{aj}}, \mathbb{G}) \\
 & & & \nearrow & & & \\
CA & \Rightarrow & \mathcal{K} & \rightarrow & \mathcal{M}_{ino} & \rightarrow & \mathcal{F}(\Gamma_{\mathcal{M}_{ino}}, \mathbb{G}) \\
 & & & \Searrow & & & \\
 & & & & \mathcal{F}(\Gamma_{\mathcal{K}}, \mathbb{G}) & &
\end{array}
$$

$\rightarrow$: encoding
$\Rightarrow$: faithful encoding

- **Remarks:**
  - $r \geq 5$ needed for "$CA \Rightarrow \mathcal{K}$"
  - $\mathbb{G} \not\subseteq \mathcal{U}$ needed for "$\mathcal{K} \Rightarrow \mathcal{F}(\Gamma_{\mathcal{K}}, \mathbb{G})$"

Assembling previous results we have:

- $d_{\mathcal{F}}(\mathcal{U}) = 1$ for the following $\mathcal{F}$:
  - $\mathcal{K}$, any unambiguous $\mathcal{F}(\Gamma_{\mathcal{K}}, \mathbb{G})$
  - $\mathcal{M}_{aj}$, any unambiguous $\mathcal{F}(\Gamma_{\mathcal{M}_{aj}}, \mathbb{G})$
  - $\mathcal{M}_{ino}$, any unambiguous $\mathcal{F}(\Gamma_{\mathcal{M}_{ino}}, \mathbb{G})$

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  - $\mathcal{M}_{ino}$, any unambiguous $\mathcal{F}(\Gamma_{\mathcal{M}_{ino}}, \mathbb{G})$
- for $\mathcal{F} = \mathcal{K}$ or $\mathcal{F} = \mathcal{F}(\Gamma_{\mathcal{K}}, \mathbb{G})$:
  - $\mathcal{U}$ is undecidable
  - ...

# Captive CA and reversibility

**Proposition**

*For any RCA $F$ there exists a RCA $G$ with:*

- $F \preceq G$
- $G$ *and* $G^{-1}$ *have a common wall state* $w$

$$\forall x_{-r}, \ldots, x_r, G(x_{-r}, \ldots, x_{-1}, w, x_1, \ldots, x_r) = w$$

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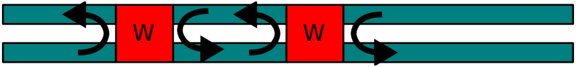
- **Idea:** define $G$ over $Q_F \times Q_F \cup \{w\}$ with



# Captive CA and reversibility

**Proposition**

*There exists a reversible-universal captive CA.*

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*There exists a reversible-universal captive CA.*

- **Proof sketch**
  - start from a reversible-universal CA $F$ with a wall state
  - use classical simulation of $F$ over configuration of the form

$$\cdots \# q_1 \cdots q_n \# x_0 \# q_1 \cdots q_n \# x_1 \# q_1 \cdots q_n \# x_2$$

    where $Q_F = \{q_1, \ldots, q_n\}$
  - local checking of this SFT condition
  - interpret errors as the wall state

# Unresolved problems

- faithful encoding for $\mathcal{M}_{aj}$ and $\mathcal{M}_{ino}$
- $\Gamma_F(G, H) \cap \mathcal{U}$ when $\{F, G, H\} \cap \mathcal{U} = \emptyset$?
- a reversible universal captive CA in 2D?

# Future directions

- extension to other amalgamation families
- decidability
- families for the reversible case:
  - with almost only reversible CA (2D case)?
  - with almost only reversible-universal CA?
- $\mu$-limit sets
- the case of fixed state set and increasing neighbourhood

## End of Médoc session...


# Questions?