On μ-limit Sets of Cellular Automata

Winter FRAC'12 — Créteil

G. Theyssier

LAMA (CNRS, Université de Savoie, France)

February 2012

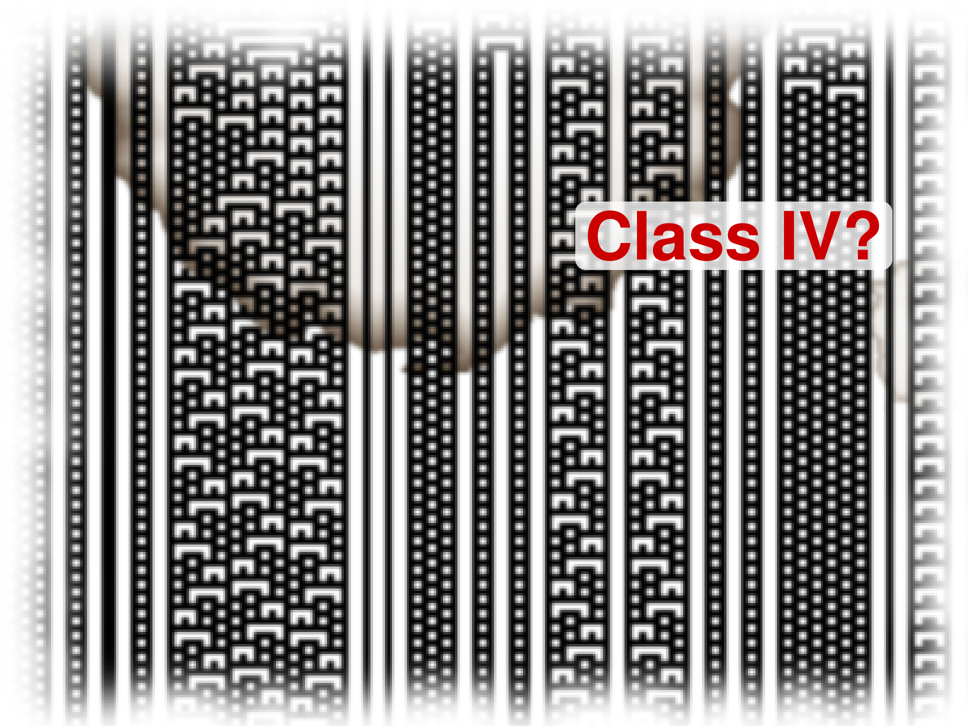
Overview of the talk

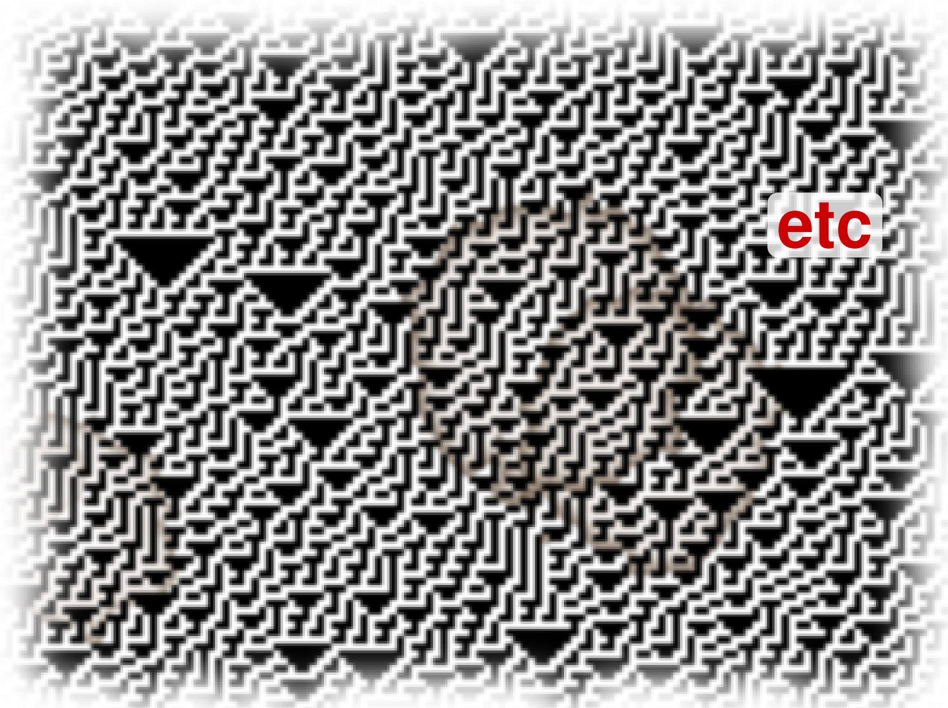
1. What and why
2. Basic Facts and Core Construction
3. (Un-)Deciding Properties of μ-limit sets
4. Future research

Overview of the talk

1. What and why
2. Basic Facts and Core Construction
3. (Un-)Deciding Properties of μ-limit sets
4. Future research


Class II?


Class IV?


etc

μ-limit set (Ω_μ)

- $[u]$: configurations where word u occurs in the center
- μ a translation invariant measure

Definition

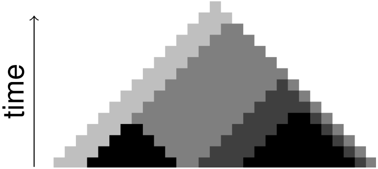
- u is a μ-limit word if

$$\lim_{t\to\infty} \mu\left(F^{-t}([u])\right) \not\to 0$$

- Ω_μ is the set of configurations made only of μ-limit words

Limit Sets of Cellular Automata Associated to Probability Measures
P. Kůrka, A. Maass, 2000

μ-limit set (Ω_μ)



- $Q = \{0, 1, 2, 3, 4\}$
- $r = 1$
- $f(x, y, z) = \mathtt{max}(x, y, z)$

- $\Omega_\mu = \{ {}^\omega 4^\omega \}$

1. $u \in (Q \setminus \{4\})^* \Rightarrow$ pre-images of u in $(Q \setminus \{4\})^*$
2. $\mu\big((Q \setminus \{4\})^n\big) \to 0$ when $n \to \infty$

Ω_μ and density

- density of word u in configuration c

$$d_u(c) = \limsup_{n\to\infty} \frac{\left|c_{-n,n}\right|_u}{2n+1}$$

- configuration c is μ-generic if $d_u(c) = \mu([u])$ for all u

Property

The following are equivalent:

1. u is a μ-limit word for F
2. for any μ-generic configuration c

$$d_u(F^t(c)) \not\to 0$$

when $t \to \infty$

Overview of the talk

1. What and why
2. **Basic Facts and Core Construction**
3. (Un-)Deciding Properties of μ-limit sets
4. Future research

Ω vs. Ω_μ

- $\Omega(F) = \bigcap_t F^t(Q^{\mathbb{Z}})$

Properties of Ω

- always contains a uniform configuration
- singleton or infinite
- $\Omega(F) = \Omega(F^k)$ for any $k \geq 1$

- $\Omega(F) = \bigcap_t F^t(Q^{\mathbb{Z}})$

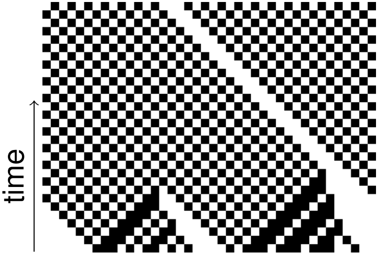
Properties of Ω

- always contains a uniform configuration
- singleton or infinite
- $\Omega(F) = \Omega(F^k)$ for any $k \geq 1$

what about Ω_μ?


$|\Omega_\mu(F)| = 2$

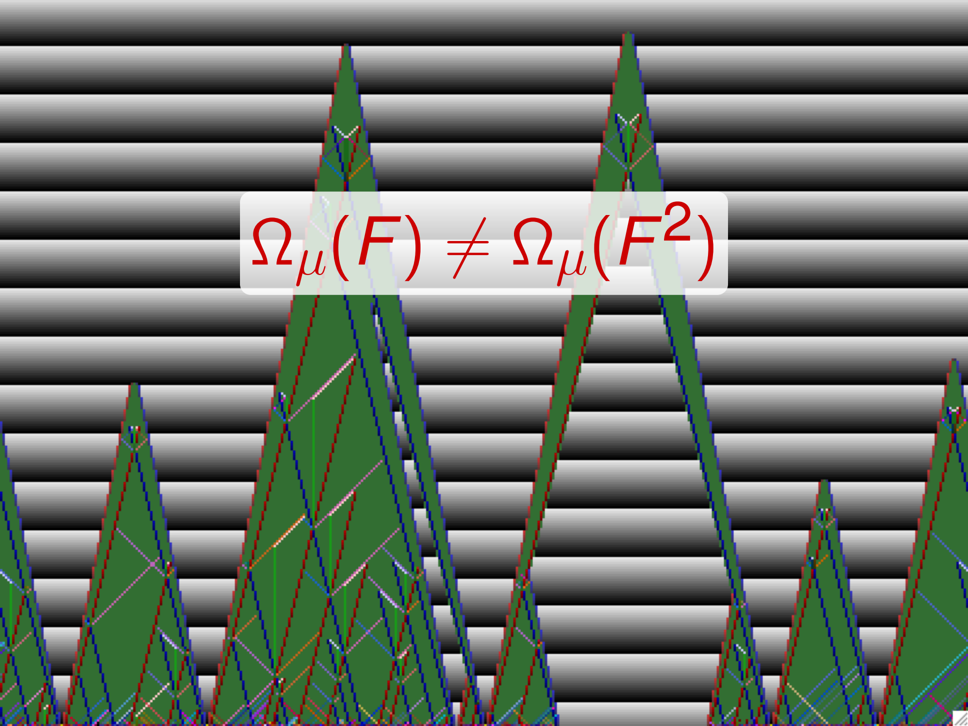
184



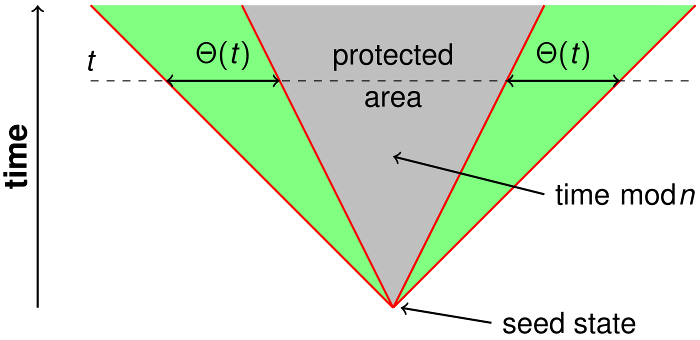
- $Q = \{0, 1\}$
- $r = 1$
- $f(x, y, z) = (x \wedge \overline{y}) \vee (y \wedge z)$

- $\Omega_\mu = \{{}^\omega 01^\omega, {}^\omega 10^\omega\}$
- Ω_μ is **not** a μ-attractor (neither Cantor, nor Besicovitch)

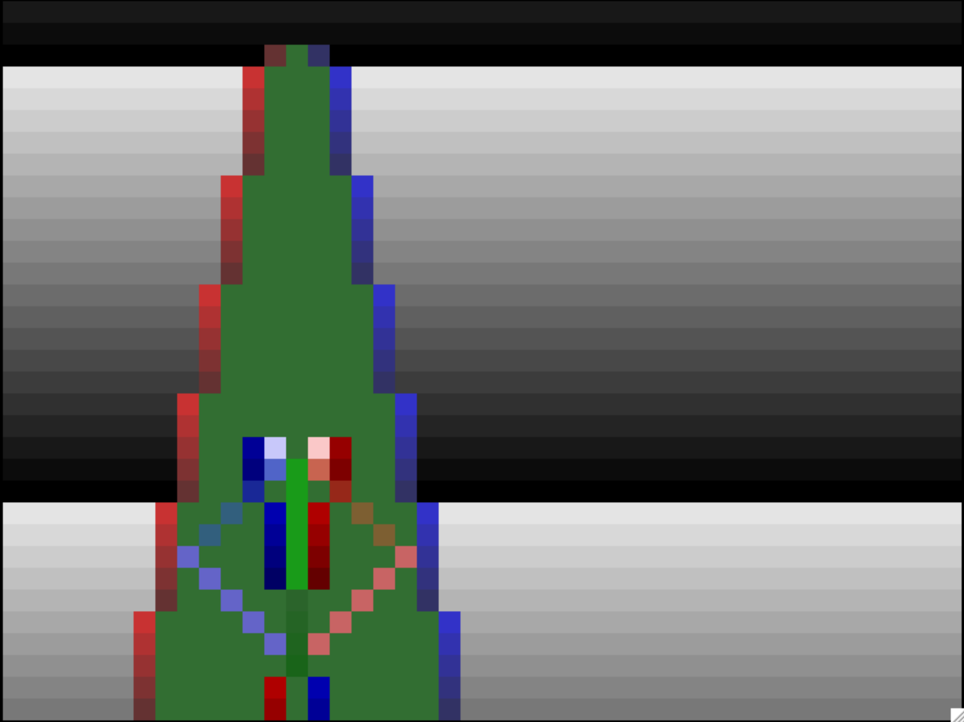
Stability of Subshifts in Cellular Automata
P. Kůrka, A. Maass, 2002

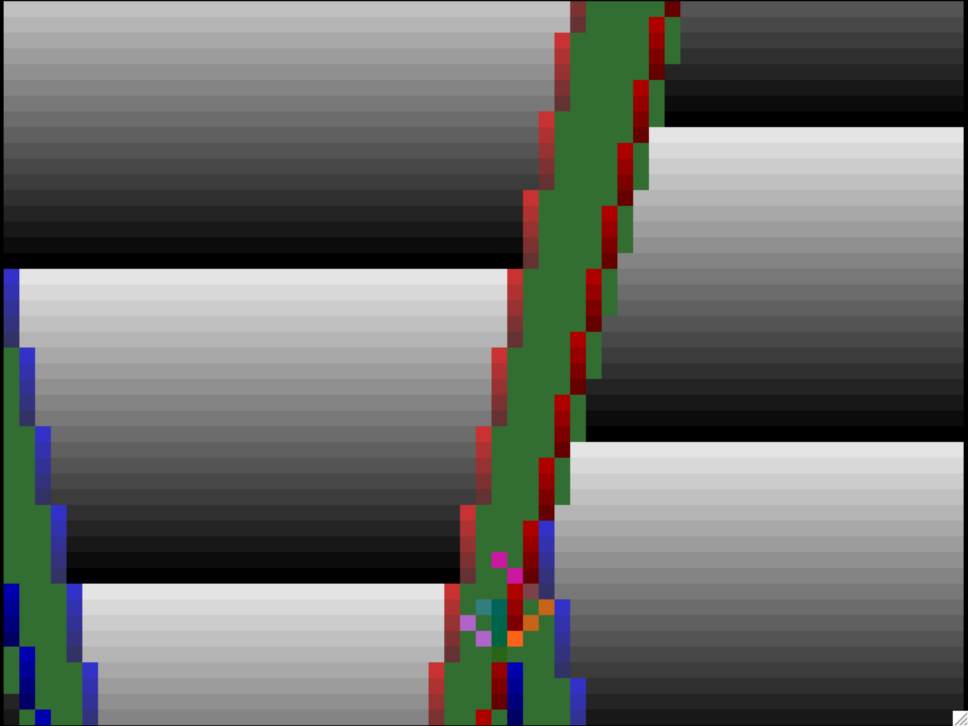

$\Omega_\mu(F) \neq \Omega_\mu(F^2)$

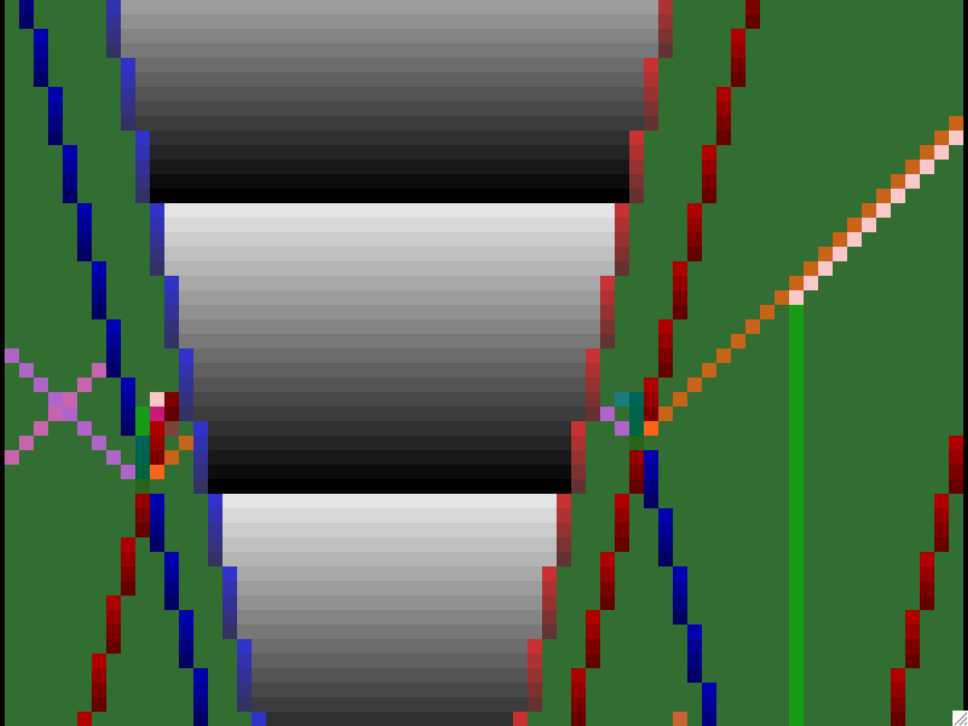
Absolute Time CA



1. only a valid zone can stop a valid zone
2. when two valid zones meet, the older is destroyed
3. two valid zones of equal age merge when they meet







Implementation details

Construction for n=20:

- 2733 states
- radius 4

Is there a significantly smaller solution?

Other property

CA with equicontinuous points but none in the image set $F(Q^{\mathbb{Z}})$

Directional Dynamics along Arbitrary Curves in Cellular Automata
M. Delacourt, V. Poupet, M. Sablik, GT, 2010

Presence of equicontinuous points

Definition

- u is a wall if $\forall t, \exists |w_t| \geq 2r$s.t.$F^t([u]) \subseteq [w_t]$;
- any word in the period of the orbit of ${}^\omega w u^\omega$ for u a wall is a **brick of wall**

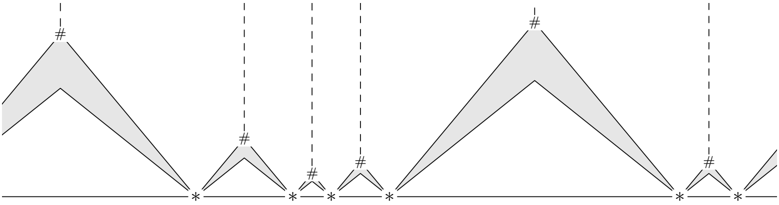
Proposition

F with equicontinuous points and μ σ-ergodic of full support

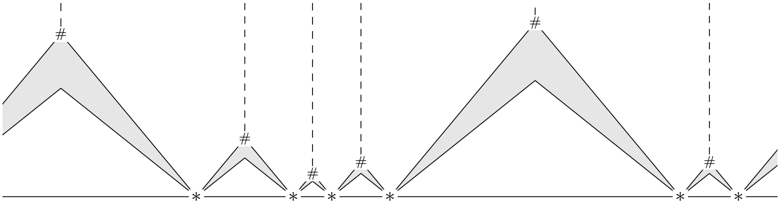
- $\Omega_\mu(F)$ is the set of brick of walls
- if F is surjective then $\Omega_\mu(F) = Q^{\mathbb{Z}}$

- **remark:** μ uniform measure and F surjective $\Rightarrow$ $\Omega_\mu(F) = Q^{\mathbb{Z}}$

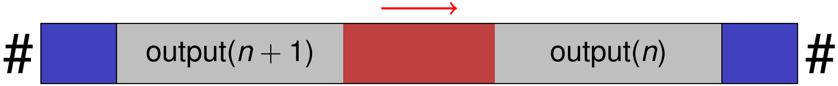
Construction Toolbox



Construction Toolbox



Computation segment:



= **computation area** (Turing head + working space)

= **merging process info** (time, length, random bits,...)

= write once **output**

Construction Toolbox

IF

- segment size $\to \infty$
- sizeof(whole segment) $>>$ sizeof(non-output part)

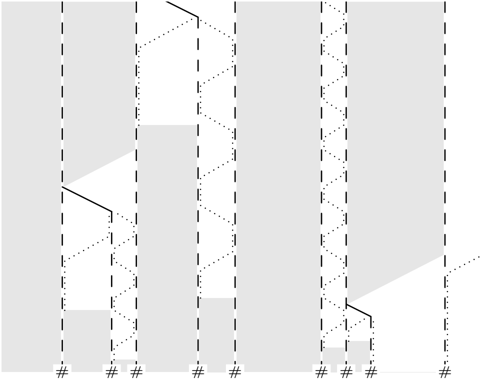
THEN

Characterization of Ω_μ

μ-limit word **are exactly** words which are persistent in the computation output:

$$\Omega_\mu = \{u : d_u(\text{output}(n)) \not\to 0\}$$

Asynchronous Merging Process



- bounded computation
- once finished ⇒ merging offer to the left, right, left, right,...
- left/right oscillation frequency depends on segment size
- deadlock possible but density=0 for μ Bernoulli

Asynchronous Merging Process

Applications

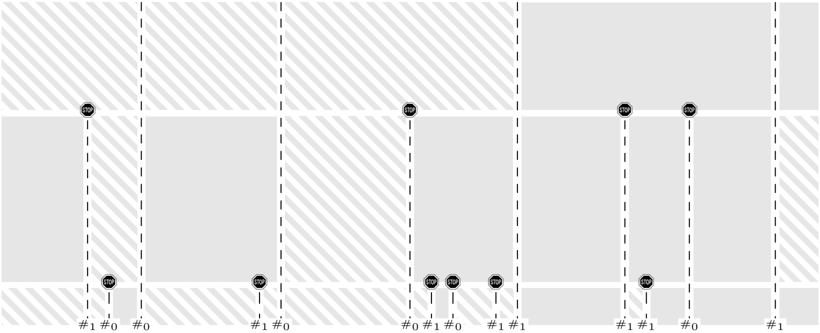
Theorem

The following subshifts can be obtained as Ω_μ:

- transitive sofic subshifts,
- substitutive subshifts (primitive substitution),
- In general any (r.e.) union of subshifts having a "generable" (recursive + conditions) generic configuration.

Construction of μ-limit sets
L. Boyer, M. Delacourt, M. Sablik, 2010

Synchronous Merging Process



- merging only at steps K^i (K fixed)
- current time step knowledge in each segment
- randomized merging policy
- merging always forced for too small segments

Synchronous Merging Process

Applications

- varying state set Q
- for each Q, $\mu =$ uniform measure
- property of Ω_μ = set of sets of configurations

Theorem

Any non-trivial property of Ω_μ is undecidable.

Rice Theorem of μ-limit Sets
of Cellular Automata
M. Delacourt, 2011

Overview of the talk

Nilpotency vs. μ-nilpotency

- F is **nilpotent** if $|\Omega(F)| = 1$
  - nilpotency is Σ_1^0-complete
  - nilpotency = easiest property of limit sets

Rice Theorem of Limit Sets
of Cellular Automata
J. Kari, 1992

Nilpotency vs. μ-nilpotency

- F is **nilpotent** if $|\Omega(F)| = 1$
  - nilpotency is Σ_1^0-complete
  - nilpotency = easiest property of limit sets

Rice Theorem of Limit Sets of Cellular Automata
J. Kari, 1992

- F is μ-**nilpotent** if $|\Omega_\mu(F)| = 1$

Theorem (M. Delacourt, Phd thesis)

μ-nilpotency reduces to any non-trivial property of Ω_μ

Degree of μ-nilpotent CA?

- μ computable if there is $f : Q^* \times \mathbb{Q} \to \mathbb{Q}$ computable s.t.

$$\forall \epsilon, \forall u, |\mu([u]) - f(u, \epsilon)| \leq \epsilon$$

Proposition

- if μ computable the set of μ-nilpotent CA is Π_3^0
- if μ ergodic of full support, the set of μ-nilpotent CA with equicontinuous points is Σ_2^0

Degree of μ-nilpotent CA?

- μ computable if there is $f : Q^* \times \mathbb{Q} \to \mathbb{Q}$ computable s.t.

$$\forall \epsilon, \forall u, |\mu([u]) - f(u, \epsilon)| \leq \epsilon$$

Proposition

- if μ computable the set of μ-nilpotent CA is Π_3^0
- if μ ergodic of full support, the set of μ-nilpotent CA with equicontinuous points is Σ_2^0

$$\forall n \, \exists t_n \, \forall t \geq t_n \; \mu(F^{-t}([q_0])) \geq 1 - \frac{1}{n}$$

Degree of μ-nilpotent CA?

- varying state set Q
- for each Q, $\mu =$ uniform measure

Proposition

The set of μ-nilpotent CA is Π_2^0-hard

Degree of μ-nilpotent CA?

- varying state set Q
- for each Q, $\mu =$ uniform measure

Proposition

The set of μ-nilpotent CA is Π^0_2-hard

- reduction from TOTAL = $\{e : \phi_e \text{ is total}\}$
- fix some e and use asynchronous merging construction
- simulate e on each successive input until no space left in segment
- $\text{output}(n) = \underbrace{1 \cdots 1}_{k} \# \underbrace{1 \cdots 1}_{k} \# \underbrace{1 \cdots 1}_{k} \# \cdots$

  where k is the last input where halt was observed
- only 1s in Ω_μ iff $e \in$ TOTAL

Corollary

Theorem

Any non-trivial property of Ω_μ is Π^0_2-hard

Overview of the talk

Future research

- complex μ-limit sets *(known: at most Σ_3^0, can be Σ_2^0-hard)*
- Cesaro mean
- Ergodic point of view
- fixed state set
- Higher dimension