Emergence of Background and Particule-like Structures in One-dimensional CA

An Attempt of Formalization

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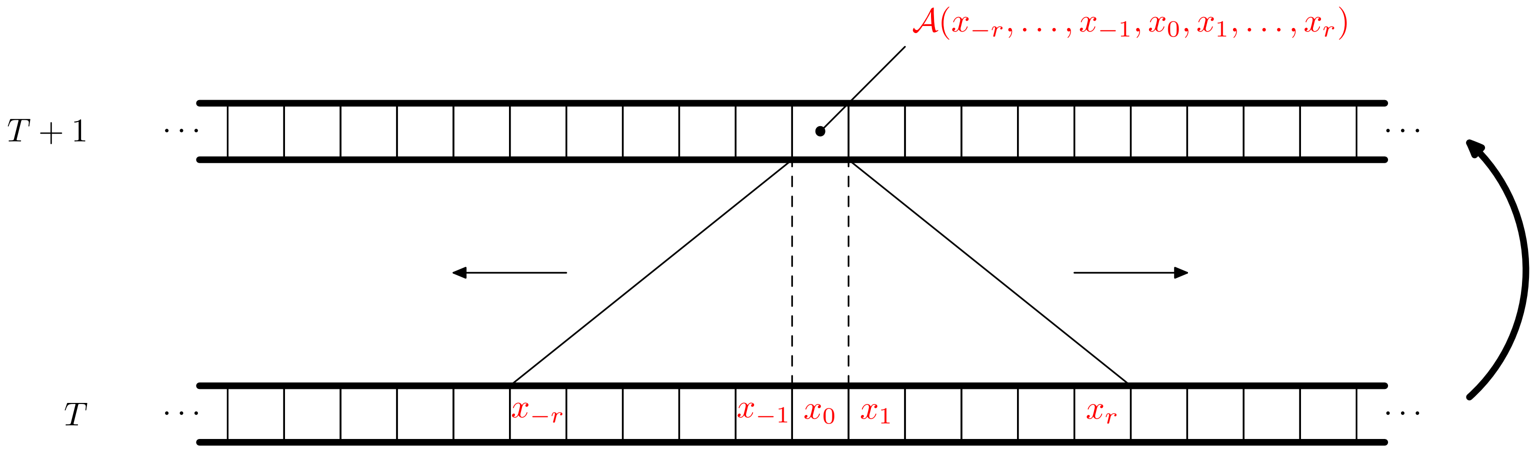
Introduction and Basic Observations

One-dimensional CA



Dual point of view

Parallel computing device

$$\mathcal{A} \,:\, A^{2r+1} \longrightarrow A$$

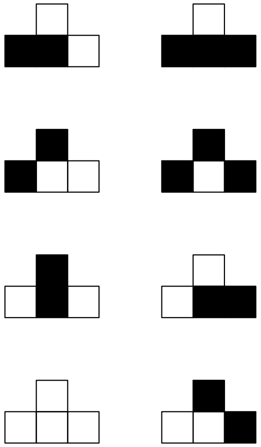
a **local** rule **uniformly** and **synchronously** applied to the line of cells

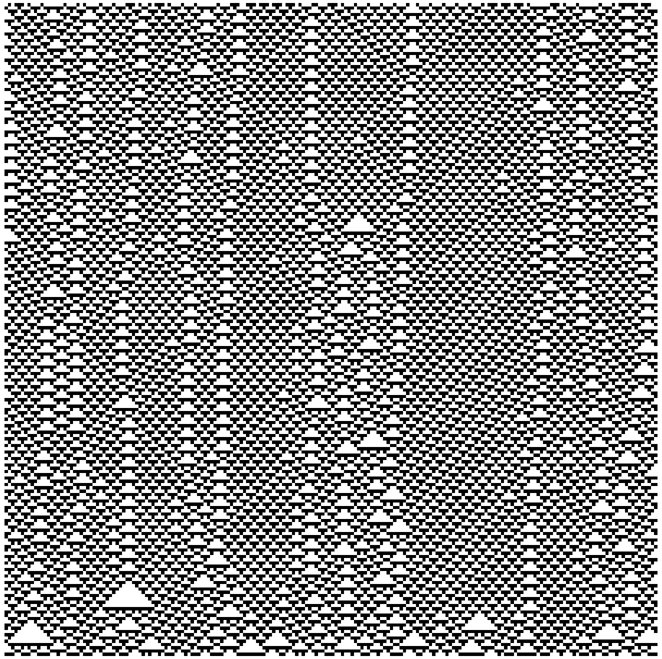
Discrete dynamical system

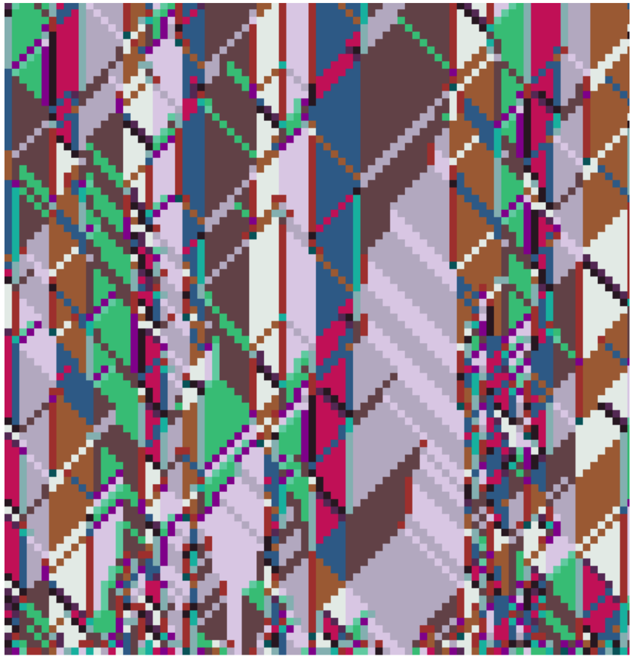
$$\mathcal{A} \,:\, A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$$

a **continuous** function which **commute with the shift** on the **compact** space $A^{\mathbb{Z}}$

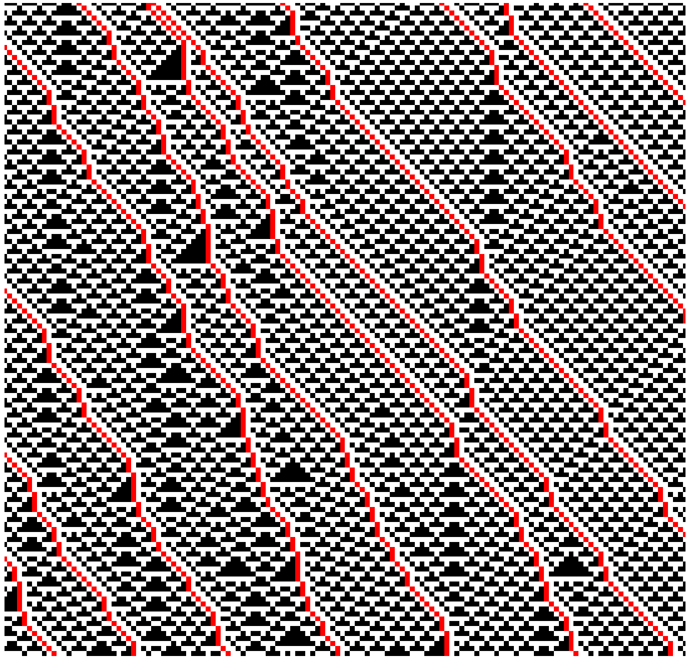
Elementary CA $W54$

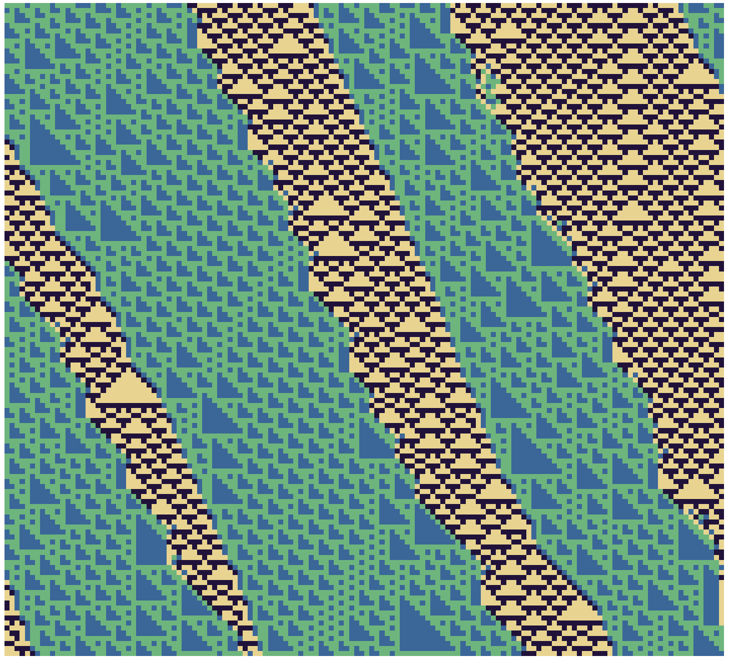






More complicated examples





Informal proposal for a generalized notion of background

Up to **re-scaling**, a **sub-automaton** which is ultimately **active** on a **sizeable part** of the line of cells.

Preliminary Notions

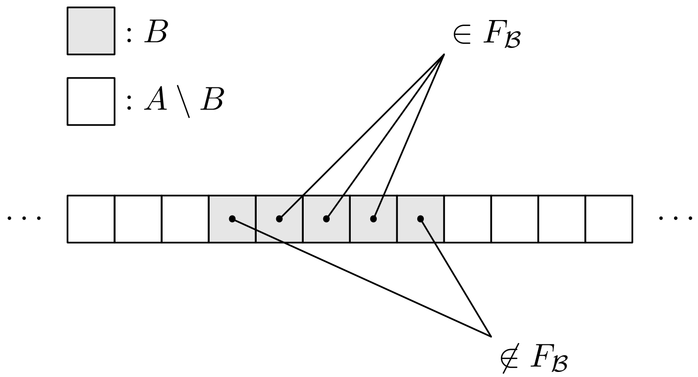
Sub-automata and "activity"

Definition :

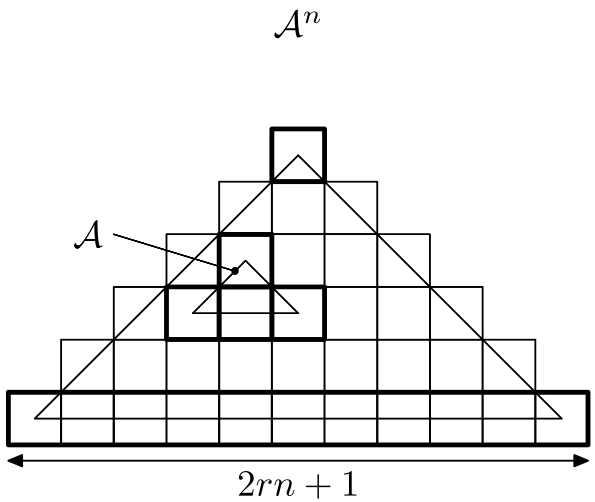
$\mathcal{B}$ is a **sub-automaton** of $\mathcal{A}$ (written $\mathcal{B} \subseteq \mathcal{A}$) if there is an injection $\phi : B \to A$ s.t. $\phi \circ \mathcal{B} = \mathcal{A} \circ \phi$ on B

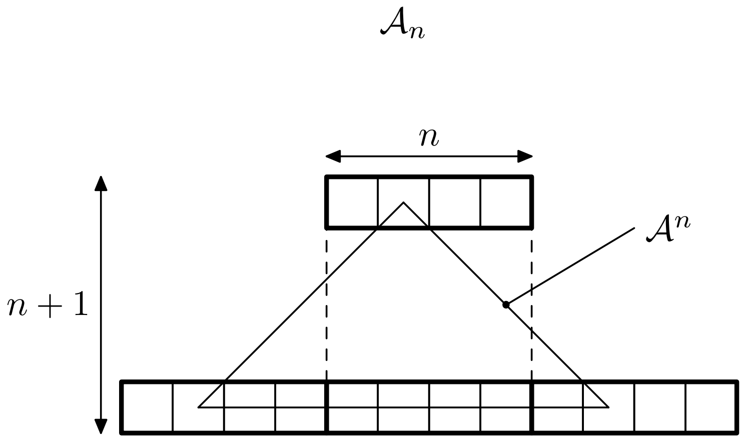
Definition :

Let $\mathcal{B} \subseteq \mathcal{A}$ and $c \in A^{\mathbb{Z}}$. The set of **active sites** of $\mathcal{B}$ in c is defined as $F_{\mathcal{B}}(c) = \{i \ : \ \forall j, |j| \leq r, \ c_{i+j} \in B\}$



Re-scaling and grouping operation





Probabilistic tools

$A^{\mathbb{Z}}$ is endowed with the **uniform product measure** : μ.

Definition :

The **density** of $X \subseteq \mathbb{Z}$ is defined as

$$d(X) = \limsup_{n\to\infty} \frac{|X \cap \{-n, \ldots, n\}|}{2n+1}$$

The Notion of α-Background

Formal definition

Definition :

For $0 \leq \alpha \leq 1$, a CA $\mathcal{B}$ is an **α-background** for a CA $\mathcal{A}$ on $\mathcal{D}$ (set of initial confi gurations) if $\exists n \in \mathbb{N}_+$ s.t. :

- $\mathcal{B} \subsetneq \mathcal{A}_n$;
- $\forall c \in \mathcal{D}, \exists t_0 \in \mathbb{N}_+$ s.t. $\forall t \geq t_0 : d\big(F_{\mathcal{B}}\big(\mathcal{A}_n^t(c)\big)\big) \geq \alpha$.

An α-background is said to be **visible** if $\mathcal{D}$ has a strictly positive measure.

About visibility

Definition :

Let σ be the **shift** automaton $\sigma : c \mapsto c'$ with $c'(i) = c(i-1)$

Theorem :

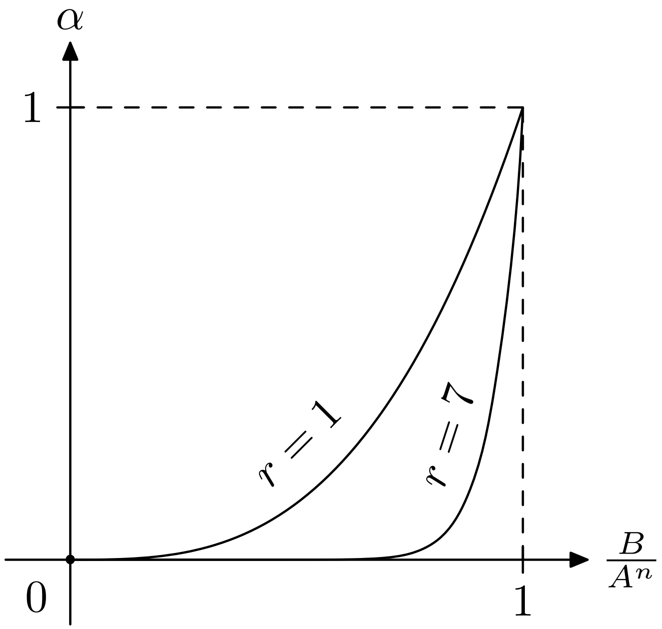
σ is ergodic (*i.e.* any measurable set closed under σ is either of measure 1 or of measure 0).

Corollary :

visible α-background $\iff \mu(\mathcal{D}) = 1$.

First Properties

Density and surjectivity



Lemma :

If $\mathcal{A}$ is **surjective** :

$\mathcal{B}$ α-background of $\mathcal{A}$ of size n implies $\alpha = \left(\frac{|B|}{|A|^n}\right)^{2r+1}$

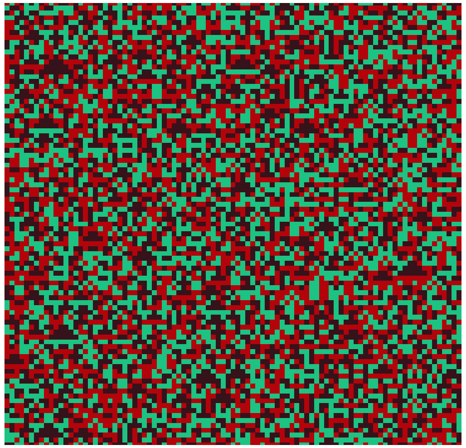
« Absence » of α-background

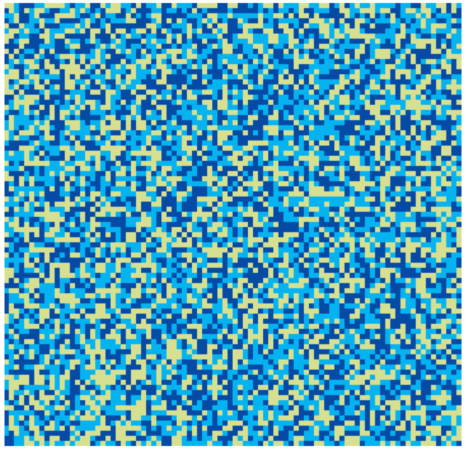
Lemma :

An additive CA $\mathcal{A}$ of radius r defined on $\mathbb{Z}_p$ (with p prime $p|2r+1$) is such that $\mathcal{B} \subseteq \mathcal{A}_n \Rightarrow \frac{|B|}{|A|^n} \leq \frac{1}{p}$.

Corollary :

For arbitrarily small α there is some CA with no visible α-background.





Constructing arbitrary α-backgrounds

Theorem :

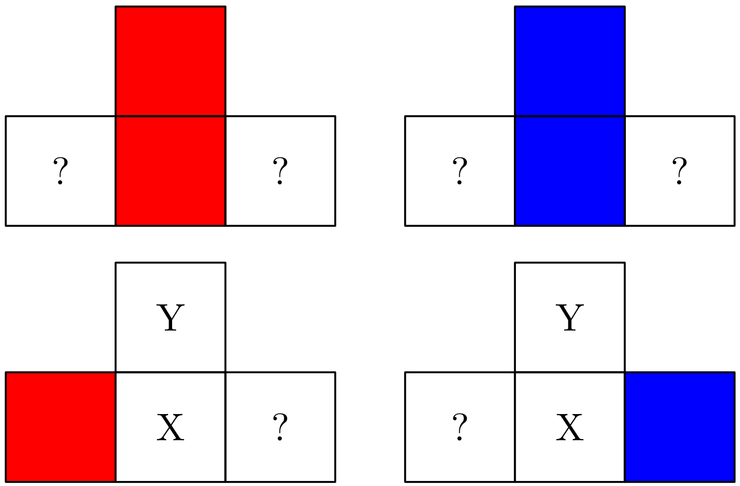
For any CA $\mathcal{A}$, $\exists$ CA $\mathcal{A}'$ s.t. $\mathcal{A} \subseteq \mathcal{A}'$ and $\forall n$: $\mathcal{B} \subseteq \mathcal{A}_n \Rightarrow \mathcal{B}$ is a visible α-background of $\mathcal{A}'$ with $\alpha > 0$.

Corollary :

$\exists$ CA with infinitely many visible α-backgrounds (each with $\alpha > 0$).

Proof sketch

$\forall n \in \mathbb{N}_+$, $\forall \mathcal{B} \subseteq \mathcal{A}_n$ build 2 wall states (left and right) such that :



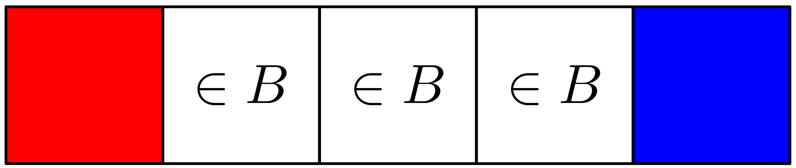
$$X \in B \Rightarrow Y \in B.$$

Properties – 5

End of proof sketch

The wall states are obtained by adding a synchronizing component to $\mathcal{A}$ (solution to the **firing squad** problem).

Then the following pattern ensures that $\mathcal{B}$ is an α-background for $\mathcal{A}$ (the active sites for $\mathcal{B}$ between the walls are preserved under iterations) :



Undecidability result

Definition :

A CA is **nilpotent** if it ultimately leads to the same stable configuration from any initial configuration.

Property :

A CA is nilpotent if and only if it admits the trivial 1-state CA as a visible 1-background.

Corollary :

It is undecidable to know if, given $\mathcal{A}$ and $\mathcal{B}$ and α, $\mathcal{B}$ is a visible α-background of $\mathcal{A}$.