On the complexity of freezing automata networks of bounded pathwidth

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Freezing cellular automata

- global map $F : Q^{\mathbb{Z}^d} \to Q^{\mathbb{Z}^d}$
- Q endowed with some order $\leq$
- **freezing property:** $\forall c \in Q^{\mathbb{Z}^d}, \forall z \in \mathbb{Z}^d : F(c)_z \leq c_z$

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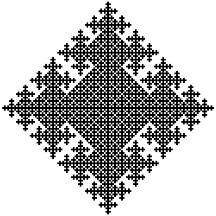
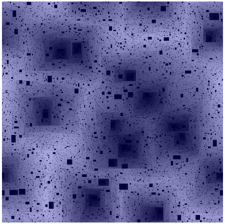


Ulam's rule 1



Bootstrap percolation

Life without death

Complexity of freezing CA

Theorem (Ollinger-Theyssier,2021)

	1D freezing CA	**2D** freezing CA
nilpotency	decidable	undecidable
prediction	NL	P-complete
trace	undecidable	undecidable

Complexity of freezing CA

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- **nilpotency:** *all orbits converge to the same fixed point*
- **prediction problem:** *given finite init. conf. and time t what is the value of cell 0 at time t?*
- **trace problem:** *given finite patterns u and v, is there an orbit from $[u]$ to $[v]$?*

$$[u] = \{c \in Q^{\mathbb{Z}^d} : c_{|D} = u\} \text{ where } u : D \subseteq \mathbb{Z}^d \to Q$$

► *Blondel-Delevenne-Kůrka's universality in dynamical systems*

Freezing automata networks

- $G = (V, E)$ a graph
- local maps: $\delta_v : Q^{N^-(v)} \to Q$ / global map: $F : Q^V \to Q^V$
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Theorem (Goles-Montealegre-Rios-Theyssier, 2021)

- freezing AN of **bounded degree** and **bounded treewidth** have a NC trace specification problem
- hardness results otherwise

- **trace specification problem:** *given a set of allowed traces at each node v, is there an orbit such that the trace at each node is allowed?*
- nilpotency/prediction/reachability reduce to trace problem

Trace properties vs. logic on orbits

Fact

Nilpotency can be LOGSPACE-reduced to **trace specification** for freezing CA.

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- another way to see nilpotency:

$$\exists x : x \to x \wedge \forall y, y \to^{+} x$$

- $c \to d$: configuration d reached from c in *one* step
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- **FO^{+}: first order logic with predicates $\to$ and $\to^{+}$**

Finite vs. infinite 1D freezing

- finite "1D" AN $\equiv$ bounded degree and bounded pathwidth
- recap of complexity results

	Infinite 1D CA	Finite "1D" AN
Nilpotency	Undecidable / Decidable	PSPACE-complete / ?
Trace properties	Undecidable / Undecidable	PSPACE-comp. / ?
FO^+	Undecidable / ?	PSPACE-complete / ?

(*how to read the table:* general case / freezing case)

Finite vs. infinite 1D freezing

- finite “1D” AN $\equiv$ bounded degree and bounded pathwidth
- recap of complexity results + **our contributions:**

	Infinite 1D CA	Finite “1D” AN
Nilpotency	Undecidable / Decidable	PSPACE-complete / **co-NL**
Trace properties	Undecidable / Undecidable	PSPACE-comp. / **NL-comp.**
FO^+	Undecidable / ?	PSPACE-complete / **NP-hard**

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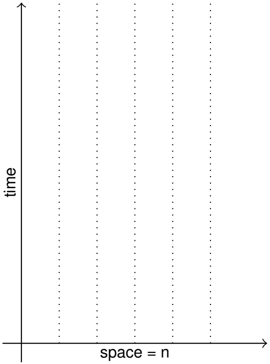
1 minute proof sketch

Trace properties are NL



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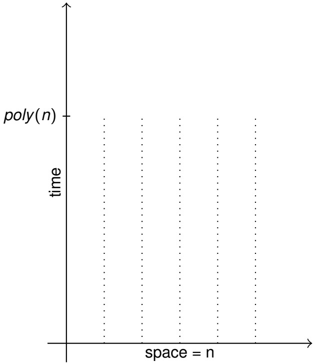

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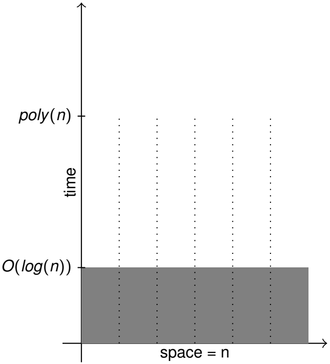

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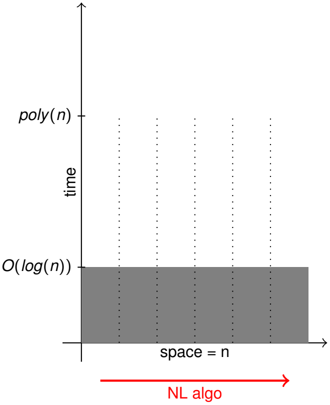

- **question:** ∃ orbit with specified traces?
- convergence of orbits in poly time
- succinct representation of traces
- NL algorithm:
  - guess traces from left to right
  - check adjacent traces

1 minute proof sketch

FO^+ is NP-hard

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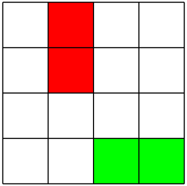
FO^+ is NP-hard

- reduction from 2D tiling problem



1 minute proof sketch

FO^{+} is NP-hard

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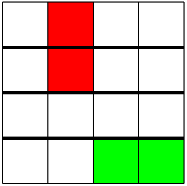

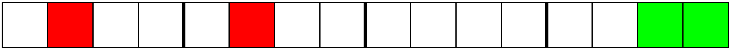

1 minute proof sketch

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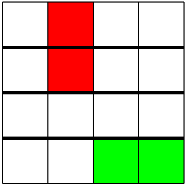
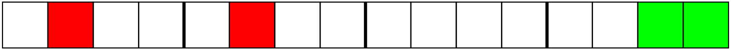
- 1 valid orbit = check 1 vertical domino





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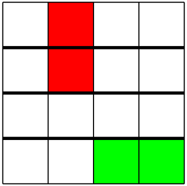

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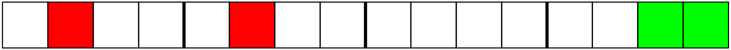

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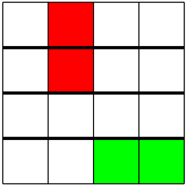

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- **technical:** FO^+ characterization of validity