Automata networks

3 short stories

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Overview

Finite and
multicomponent
dynamical system

Distributed
computational device
with **bounded memory**

- McCulloc and Pitts (1940s)
- gene interaction networks
- social interaction networks
- distributed computing, graph automata
- memoryless computation
- network coding

Plan

1. Finite maps
2. Labeled graphs
3. Succinct graphs

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1. **Finite maps**
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Finite maps

- Q finite alphabet

Definition

An automata network is a **map** $F : Q^n \to Q^n$ for some n.

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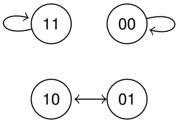
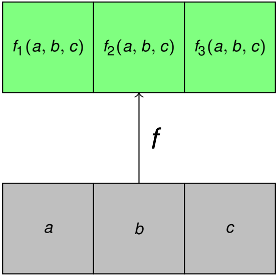
- **Example**: swapping Boolean registers

$$\sigma : \{0,1\}^2 \to \{0,1\}^2$$
$$\sigma(a,b) = (b,a)$$



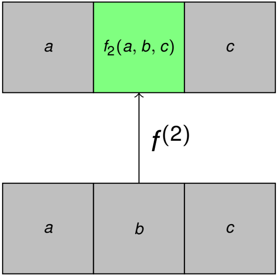
Instructions, sequentialization

- $f : Q^n \rightarrow Q^n$



Instructions, sequentialization

- $f : Q^n \to Q^n$
- $f^{(v)} : Q^n \to Q^n$
- $f^{(v)}(x)_i = \begin{cases} f(x)_i & \text{if } i = v \\ x_i & \text{else.} \end{cases}$



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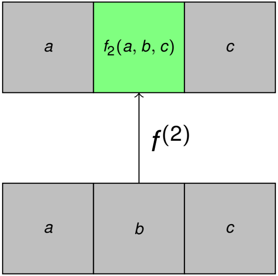

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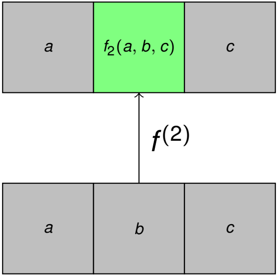

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- $\sigma(a, b) = (b, a)$
- $\langle \sigma \rangle_{\mathrm{Seq}} = ?$

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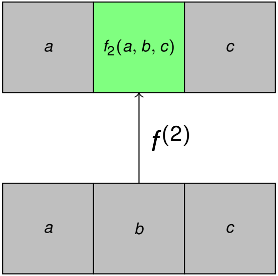

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- $\sigma(a, b) = (b, a)$
- $\langle \sigma \rangle_{\mathrm{Seq}} = \{id; \sigma^{(1)}; \sigma^{(2)}\}$

What map can be sequentialized?

- say g is **sequentialized** by f if $g \in \langle f \rangle_{\mathrm{Seq}}$
- can $\sigma(a, b) = (b, a)$ be sequentialized by some f?

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- yes!
  - $\oplus$: addition mod 2
  - $f(a, b) = (a \oplus b, a \oplus b)$
  - $(a, b) \stackrel{f^{(1)}}{\to} (a \oplus b, b) \stackrel{f^{(2)}}{\to} (a \oplus b, a) \stackrel{f^{(1)}}{\to} (b, a)$

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- so any $(a_1, \ldots, a_n) \mapsto (a_{\pi(1)}, \ldots, a_{\pi(n)})$.

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Theorem (Cameron-Fairbairn-Gadouleau, 2014)

$\exists f \in F(Q^n)$ such that $B(Q^n) \subseteq \langle f \rangle_{\mathrm{Seq}}$ (unless $n = |Q| = 2$).

- $B(Q^n)$: bijections $Q^n \to Q^n$
- $F(Q^n)$: maps $Q^n \to Q^n$

What map can be sequentialized?

- the following $g \in F(\{0,1\}^2)$ is **not** sequentializable:

$$00 \mapsto 01 \mapsto 11 \mapsto 10 \mapsto 00$$

- **F. Bridoux** by computer search: any $g \in F(\{0,1\}^3)$ is sequentializable

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Theorem (Bridoux-Gadouleau-Theyssier, 202?)

For any $n \geq 5$ **any** $g \in F(\{0,1\}^n)$ **is** sequentializable.

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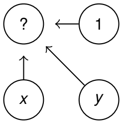

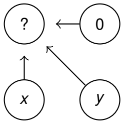

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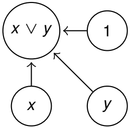

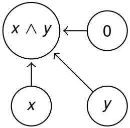

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- majority networks are **universal**, in particular:
  - $\exists$ transients and cycles of **exponential** length
  - reachability ($y \in Orbit(x)$?) is **PSPACE-complete**

The impact of symmetry

- what about majority on **unoriented** graphs?

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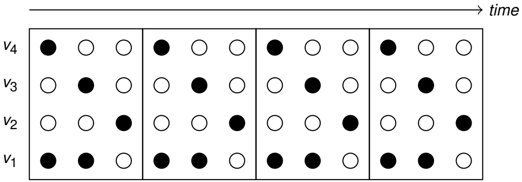
- much more is known (threshold rules, infinite graphs, etc)

Symmetry versus asynchronism

- asynchronism = periodic sequence of nodes updates



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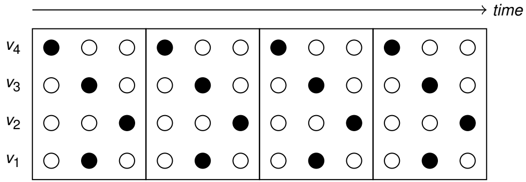

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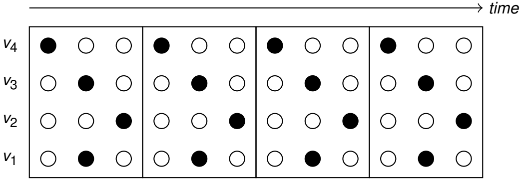

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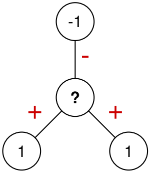
Theorem (Goles-Montealegre-Salo-Törmä, 2016)

Undirected majority networks under **bloc sequential** schedules are **universal.**

- NB: sequence of updates of constant period is enough

On the edge of universality

- $Q = \{-1, 1\}$
- ponderation $\pm$ on edges
- $\delta_v =$ ponderated minimum



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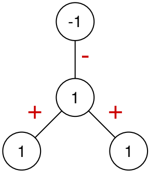

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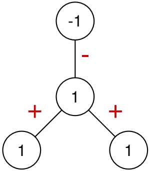
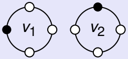
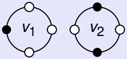
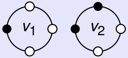
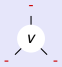
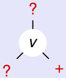
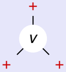


Theorem (Ríos Wilson-Theyssier, 202?)

	synchronous	 bloc sequential	 local clocks	 periodic
				
				
				

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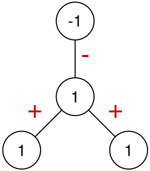

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	synchronous	v_1 v_2 bloc sequential	v_1 v_2 local clocks	v_1 v_2 periodic
- v - -	$\leq$ Cst $\leq$ Cst NC_0	**universal**	**universal**	**universal**
? v ? +	$\leq$ poly $\leq$ poly PTIME	$\leq$ poly $\leq$ poly PTIME	**universal**	**universal**
+ v + +	$\leq$ poly $\leq$ poly NP	$\leq$ poly $\leq$ poly NP	$\leq$ poly $\leq$ poly NP	$\leq$ poly **superpoly** NP

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Orbit graphs

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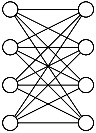
Definition

A **non-deterministic** automata network is a **directed graph** represented **succinctly** by a Boolean circuit $C(x, y)$ computing its adjacency relation.

- vertices are identified by labels: $x, y \in \{0, 1\}^n$
- size of the graph: $N \leq 2^n$



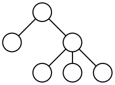
- $N = 2^n$
- $C(x, y) = \begin{cases} 1 & \text{if } x_1 \neq y_1, \\ 0 & \text{else.} \end{cases}$

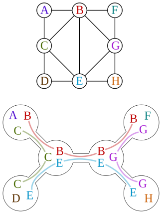
Recall: Courcelle's theorem

Theorem (Courcelle,1990)

Any MSO formula can be tested in linear time on graphs of bounded treewidth.

- treewidth:





- monadic second order logic (MSO), *e.g.* 3-colorability:

$$\exists X_1, X_2, X_3, \forall v, \left(\bigvee_i v \in X_i\right) \wedge \left(\forall v', \bigwedge_i \neg(adj(v, v') \wedge v \in X_i \wedge v' \in X_i)\right)$$

Succinct graph property testing

- For some property $\mathcal{P}$ of graphs:
  - **input**: N and circuit $C(x, y)$
  - **question**: does the graph $G_{N,C}$ satisfy $\mathcal{P}$?

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Theorem (Gammard-Guillon-Perrot-Theysier, 202?)

Any non-trivial MSO property on **bounded treewidth succinct** graphs is either NP-hard or co-NP-hard.

- non-trivial $\equiv \infty$ models and ∞ counter-models of treewidth $\leq k$
- Remark: determinism $\implies$ bounded treewidth orbit graph

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Theorem (Gammard-Guillon-Perrot-Theysier, 202?)

There are non-trivial MSO property on **succinct** graphs which are **neither** NP-hard **nor** co-NP-hard (under reasonable complexity assumption).

- non-trivial $\equiv \infty$ models and ∞ counter-models

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