Cellular Automata and Boostrap Percolation

SDA2 @ Liège

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June 2022

The cave of SDA2

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General CA on random configurations

Bootstrap percolation

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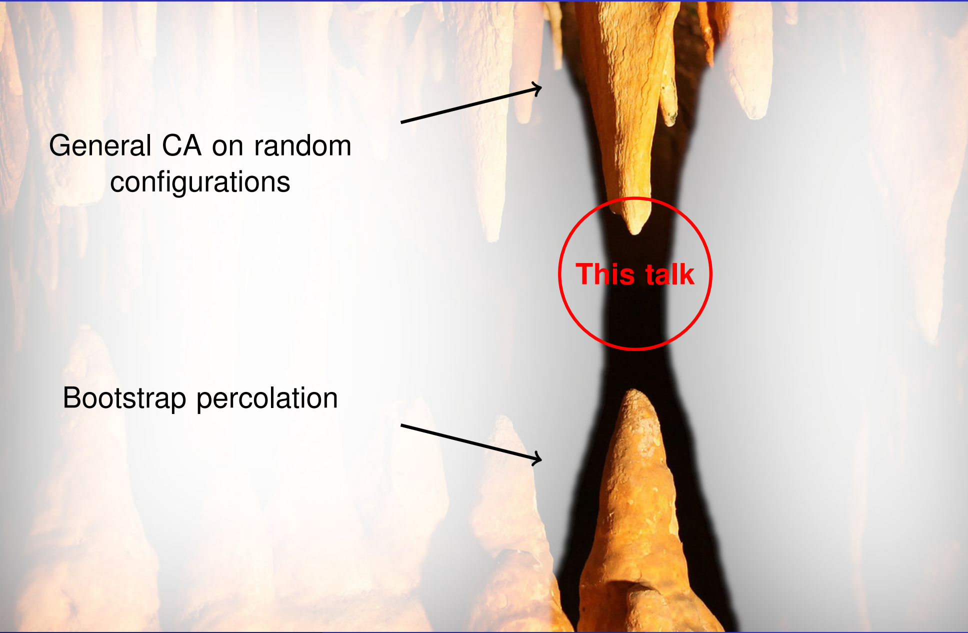

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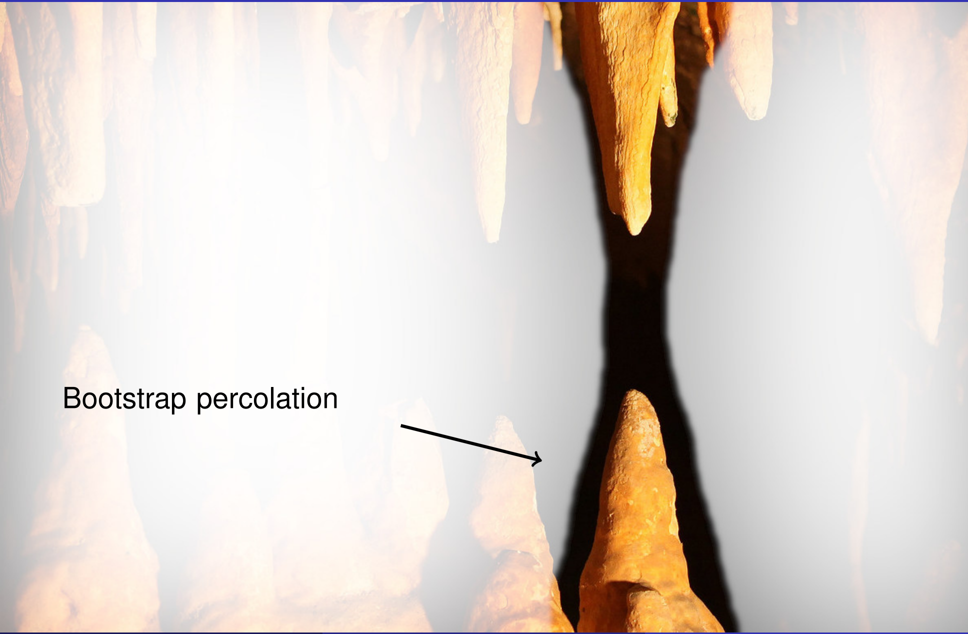

Percolation

- some graph (here $\mathbb{Z}^2$)
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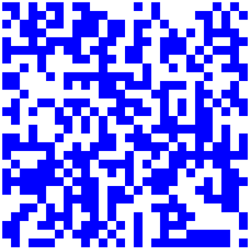

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$p = 0.6$

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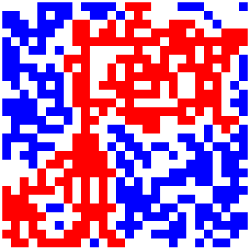

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Infinite with prob. > 0

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$p = 0.2$	$p = 0.5$	$p = 0.6$
Finite with prob. 1	**Finite** with prob. 1	**Infinite** with prob. > 0

- Open: what is the exact critical value for p?

Bootstrap percolation

Threshold models (Chalupa-Leath-Reich, 1979)

- some graph (here $\mathbb{Z}^d$), some **threshold** θ
- $E_0 \subseteq \mathbb{Z}^d$ a μ_p-random initial set

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- **invasion:** the event $\mathcal{I} = \left\{E_0 : E_t \to_t \mathbb{Z}^2\right\}$

Central question

How does almost sure invasion ($\mu_p(\mathcal{I}) = 1$) depends on p?

(many other interesting questions not in this talk)

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- **monotonicity:** $p \leq p' \implies \mu_p(\mathcal{I}) \leq \mu_{p'}(\mathcal{I})$
- **ergodicity:** $\mu_p(\mathcal{I}) \in \{0, 1\}$
- **critical probability:** $p_c = \inf\left\{p : \mu_p(\mathcal{I}) = 1\right\}$

$d = 2, \theta = 1 \implies p_c = 0$

super-critical

$$\begin{array}{ccc} & \uparrow & \\ \leftarrow & 1 & \rightarrow \\ & \downarrow & \end{array}$$

$d = 2, \theta = 3 \implies p_c = 1$

sub-critical

$$\boxed{\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}}$$

Bootstrap percolation

Threshold models (Chalupa-Leath-Reich, 1979) — examples

$d = 2, \theta = 1 \implies p_c = 0$
super-critical

$$\begin{array}{ccc} & \uparrow & \\ \leftarrow & 1 & \rightarrow \\ & \downarrow & \end{array}$$

$d = 2, \theta = 3 \implies p_c = 1$
sub-critical

$$\boxed{\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}}$$

Theorem (vanEnter-1987,Schonmann-1992)

In dimension d, if $\theta \leq d$ then $p_c = 0$

- $d = 2, \theta = 2$, $n \times n$ torus (Aizenman and Lebowitz, 1988):

$$p_c(n) \sim \frac{\lambda}{\log(n)}$$

- **physics** (Adler-Stauffer-Aharony-1989): $\lambda = 0.245 \pm 0.015$
- **mathematics** (Holroyd-2003): $\lambda = \pi^2/18 \approx 0.548311$

2D Bootstrap percolation

$\mathcal{U}$-bootstrap percolation (Bollobás-Smith-Uzzell-2015)

- **update triggers:** finite set $\mathcal{U}$ of finite sets $U_j \subseteq \mathbb{Z}^2$
- **dynamics:** $E_{t+1} = E_t \cup \{z : z + U_j \subseteq E_t \text{ for some } j\}$

- e.g.: threshold model $\theta \iff \mathcal{U} =$ subsets of size θ of N,S,E,W

- still monotonicity, **critical probability** $p_c(\mathcal{U})$

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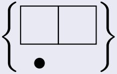
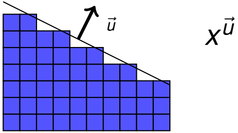
Example with $0 < p_c(\mathcal{U}) < 1$

- $\mathcal{U} =$  $= \{\{(0,1),(1,1)\}\}$
- equivalent to directed percolation on $\mathbb{Z}^2$

2D Boostrap percolation

A decidable classification

- stable directions $S(\mathcal{U})$
  - $x^{\vec{u}}(\vec{z}) = 1 \iff \vec{u} \cdot \vec{z} < 0$
  - $\vec{u} \in S(\mathcal{U}) \iff F(x^{\vec{u}}) = x^{\vec{u}}$



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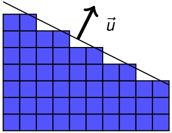


$x^{\vec{u}}$

Theorem (Bollobás-Smith-Uzzell-2015 + Balister-Bollobás-Przykucki-Smith-2016)

- $S(\mathcal{U})$ is computable from $\mathcal{U}$ (finite union of closed rational intervals)
- $p_c(\mathcal{U}) = 1 \iff S(\mathcal{U}) = S^1$
- $p_c(\mathcal{U}) = 0 \iff$ open semi-circle disjoint from interior of $S(\mathcal{U})$

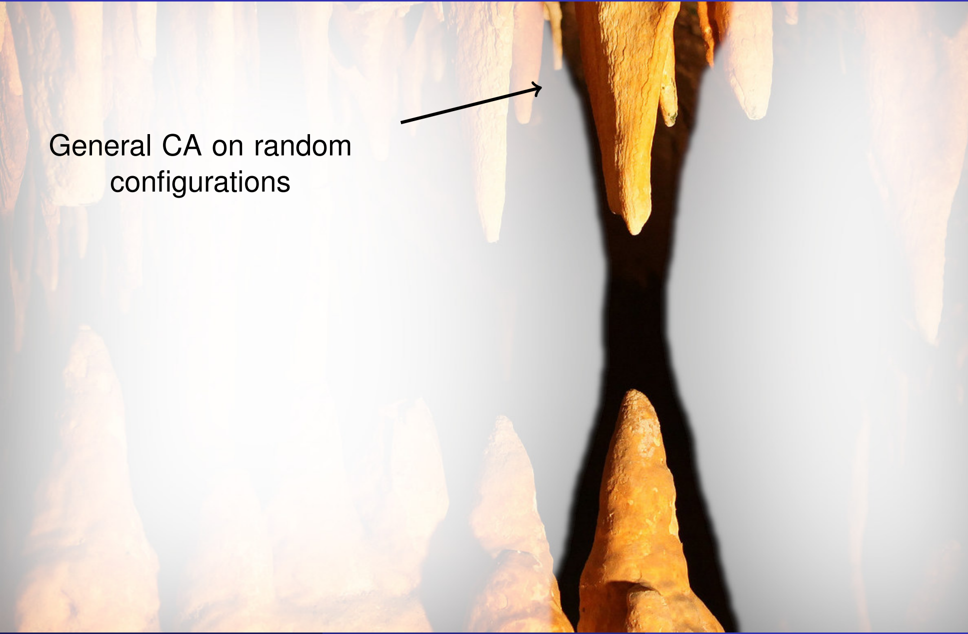
$\theta = 1$, $p_c = 0$ | $\theta = 3$, $p_c = 1$ | $\theta = 2$, $p_c = 0$ | $\mathcal{U} = \{\{(0,1),(1,1)\}\}$, $0 < p_c < 1$

Boostrap percolation

It was only the tip of the iceberg

- results on finite size $n \times n$ and $p_c(n)$
- **rigidity** (“Universality”): 3 possible behaviors determined by $S(\mathcal{U})$
- asymptotics on $p_c(n)$ up to a constant factor for critical case (Bollobás-Duminil-Copin-Morris-Smith-2020)
- very recent (partial) extensions to **higher-dimensional** case (Balister-Bollobás-Morris-Smith-2022)
- **many other questions**: percolation speed, behavior at p_c, limit density of 1s when no invasion, etc

The cave of SDA2



- $N \subseteq \mathbb{Z}^d$ finite neighborhood, Q alphabet, $\delta : Q^N \to Q$ local rule
- deterministic **cellular automaton:** $F : Q^{\mathbb{Z}^d} \to Q^{\mathbb{Z}^d}$ with

$$F(x)_z = \delta(z' \in N \mapsto x_{z'+z})$$

- **cylinders**: for $u \in Q^D$, $[u] = \{x : x_z = u_z, \forall z \in D\}$
- μ translation-invariant Borel probability measure
- **weak convergence**: $\mu_n \to \mu \iff \forall u, \mu_n([u]) \to \mu([u])$

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- **action on measures:** $F\mu(X) = \mu(F^{-1}(X))$
- μ, $F\mu$, $F^2\mu$, ... generally not convergent
- this talk: convergent case

General cellular automata

Action on measures

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Definition

F is μ-nilpotent if there is $q \in Q$ with $F^t\mu(q') \to 0$ for all $q' \neq q$

- μ_0: uniform product measure

Theorem (Boyer-Delacourt-Poupet-Sablik-Theyssier-2015)

Knowing whether F is μ_0-nilpotent is Π_3-hard.

- μ_0: uniform product measure

Theorem (Boyer-Delacourt-Poupet-Sablik-Theyssier-2015)

Knowing whether F is μ_0-nilpotent is Π_3-hard.

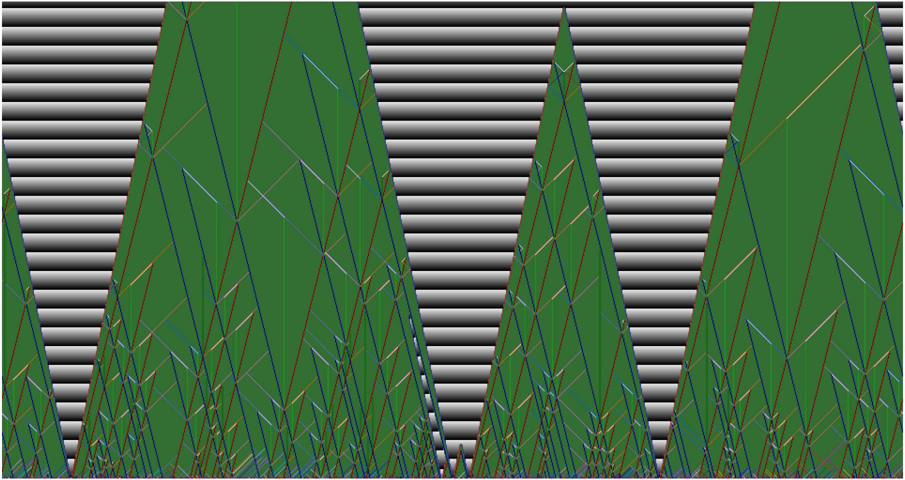
- μ **limit-computable** if $\mu([u]) = \lim_{n \to \infty} \phi(u, n)$ for ϕ computable, $\forall u$.

Theorem (1D:Hellouin-Sablik-2016/2D:Delacourt-Hellouin-2017)

$F^t \mu_0 \to \mu$ for some $F \iff \mu$ limit-computable

General cellular automata

Undecidability/realization results



Bootstrap percolation as cellular automata

- any $\mathcal{U}$-bootstrap percolation process is a CA
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Bootstrap percolation as cellular automata

- any $\mathcal{U}$-bootstrap percolation process is a CA
- what makes it special?
- order on states: $(Q, \leq) \rightsquigarrow$ order on configurations
- **freezing:** $\forall c, \forall z : c_z \leq F(c)_z$
- **monotone:** $\forall c, c' : c \leq c' \implies F(c) \leq F(c')$
- $\mathcal{U}$-bootstrap CA $\iff$ monotone freezing 2 states 2D CA

Bootstrap percolation as cellular automata

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- freezing $\implies$ convergence of $\left(F^t \mu_p\right)_t$
- monotone $\implies$ $\mu_p(\mathcal{I})$ increasing with $p \rightsquigarrow$ 1 phase transition at p_c

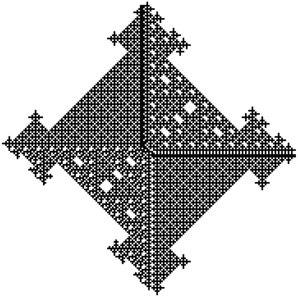
Bootstrap vs. general CA

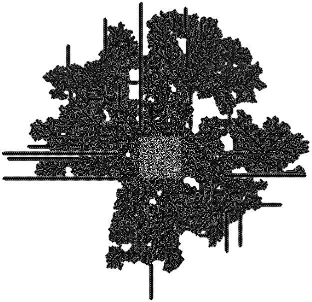
- decidability for freezing + monotone + 2-states CA
- undecidability uses non-freezing + non-monotone CA

The cave of SDA2

Freezing non-monotone CA

Classical examples



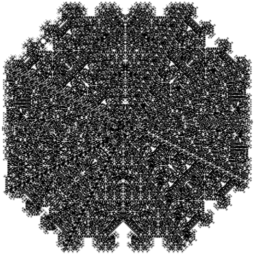


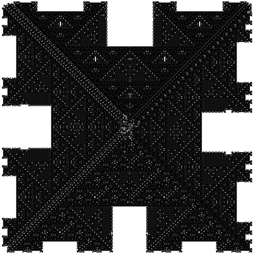
S. Ulam (1960s)
□ → ■ *if exactly 1* ■
in 4-neighborhood

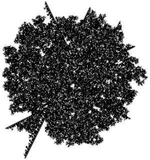
Life without death (1980s)
□ → ■ *if exactly 3* ■
in 8-neighborhood

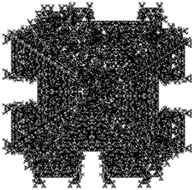
This talk: freezing non-monotone CA

Random examples









Freezing cellular automata

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Lemma

If F is freezing and μ full-support then: $F\mu$-nilpotent $\iff \mu(\mathcal{I}) = 1$.

- bootstrap percolation: how μ-nilpotency depends on μ?

Freezing cellular automata

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Theorem 0

μ-nilpotency is decidable for **1D** freezing CA and it does not depend on μ as soon as μ is ergodic full-support.

- NB: recall that 1D freezing CA are "computationally universal" (can embed Minsky machines)

Theorem 1

Given F a **2D** freezing CA with 2 states, the following properties are undecidable:

- F is μ_p-nilpotent for **some** $0 < p < 1$
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- **Lemma:** no obstacle $\iff$ μ_p-nilpotence for any large enough p
- Turing machine $M \rightsquigarrow$ freezing CA F
- M halts $\implies$ F has an obstacle
- M doesn't halt $\implies$ no obstacle + μ-nilpotent for any μ (ergo full-support).
- obstacle $\sim$ halting computation + protective shell

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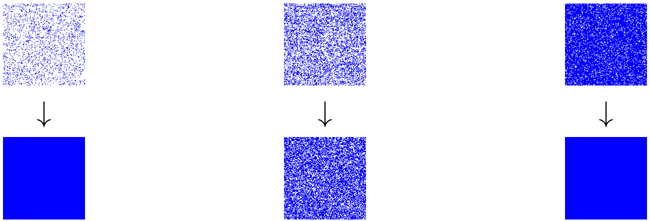
Theorem 2

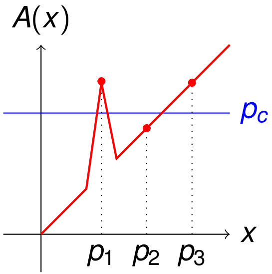
There is a 2D 2-states freezing CA F, and $0 < p_1 < p_2 < p_3 < 1$ such that F is μ_{p_1}-nilpotent and μ_{p_3}-nilpotent but not μ_{p_2}-nilpotent.

μ_{p_1}-random μ_{p_2}-random μ_{p_3}-random



Intuition:

1. $B \equiv \mathcal{U}$-**bootstrap** with $0 < p_c < 1$
2. $A \equiv$ 'non-monotone density **amplifier**'
3. F such that $F^t = B^t \circ A$



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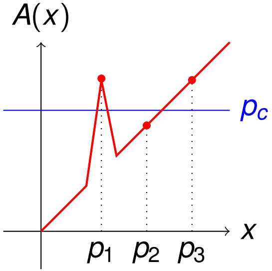

Technicalities:

- A **locally** checks that density of 1s is $\approx p_1$ (error estimate)
- $F = B \circ A +$ **block coding** and **markers** $\implies F^t = B^t \circ A$
- **lemma**: bounded correlations $\rightsquigarrow$ dominates some Bernoulli

Open Problems

- freezing monotone CA with more states?
- dimension ≥ 3?

Balister-Bollobás-Morris-Smith, in preparation

Uncomputability of critical probabilities for monotone cellular automata

[NDLR: 'monotone' means 'monotone + freezing + 2 states']

- monotone (non-freezing) CA?
- F freezing: how crazy can be the set of μ s.t. F is μ-nilpotent?
- what are the possible $\lim_{t\to\infty} F^t\mu_0$? the possible μ-limit sets?