Rice Theorems for Automata Networks and Succinct Graphs

Seminarío DISC, UAI, Santiago de Chile

G. Theyssier (joint work with G. Gamard, P. Guillon, K. Perrot)

Institut de Mathématiques de Marseille

November 11th 2021

Classical Rice Theorem

- Turing machine M computes a function $\lambda_M : D_M \subseteq \mathbb{N} \to \mathbb{N}$

Theorem (Rice, 50s)

Any non-trivial property $\mathcal{P}$ of the function computed by Turing machines is undecidable.

Classical Rice Theorem

- Turing machine M computes a function $\lambda_M : D_M \subseteq \mathbb{N} \to \mathbb{N}$

Theorem (Rice, 50s)

Any non-trivial property $\mathcal{P}$ of the function computed by Turing machines is undecidable.

- consider M_0 that never halts, so $D_{M_0} = \emptyset$
- suppose $M_0 \not\models \mathcal{P}$ and $M_1 \models \mathcal{P}$
- given any machine M define machine M^* that on input n do

$$\text{compute } M(0) \text{ ; compute } M_1(n)$$

- $M^* \models \mathcal{P}$ if and only if M halts on input 0

Automata Networks

- n nodes, alphabets Q_i, $X = \prod_{1 \leq i \leq n} Q_i$

Automata Networks

- n nodes, alphabets Q_i, $X = \prod_{1 \leq i \leq n} Q_i$
- **AN = non-deterministic** automata network: $R \subseteq X^2$
- **DAN = deterministic** AN: $F : X \to X$
- **(D)ANU = uniform alphabet** AN: $X = Q^n$
- always given by Boolean circuits

Limit sets

- **DAN**: $\Omega_F = \bigcap_t F^t(X)$ = configurations in limit cycles
- **AN**: Ω_R = configurations with unbounded history

Limit sets

- **DAN**: $\Omega_F = \bigcap_t F^t(X)$ = configurations in limit cycles
- **AN**: Ω_R = configurations with unbounded history

- $\mathcal{P}$ is a **property of limit sets** if

$$R_1 \models \mathcal{P} \Leftrightarrow R_2 \models \mathcal{P} \text{ whenever } \Omega_{R_1} = \Omega_{R_2}$$

- example: nilpotency (*i.e.* $\left|\Omega_R\right| = 1$)

Limit sets

- **DAN**: $\Omega_F = \bigcap_t F^t(X)$ = configurations in limit cycles
- **AN**: Ω_R = configurations with unbounded history

- $\mathcal{P}$ is a **property of limit sets** if

$$R_1 \models \mathcal{P} \Leftrightarrow R_2 \models \mathcal{P} \text{ whenever } \Omega_{R_1} = \Omega_{R_2}$$

- example: nilpotency (*i.e.* $\left|\Omega_R\right| = 1$)

- decision problem for property $\mathcal{P}$:
  - **input:** ANU R
  - **question:** $R \models \mathcal{P}$?

Limit sets

- **DAN**: $\Omega_F = \bigcap_t F^t(X)$ = configurations in limit cycles
- **AN**: Ω_R = configurations with unbounded history

- $\mathcal{P}$ is a **property of limit sets** if

$$R_1 \models \mathcal{P} \Leftrightarrow R_2 \models \mathcal{P} \text{ whenever } \Omega_{R_1} = \Omega_{R_2}$$

- example: nilpotency (*i.e.* $\left|\Omega_R\right| = 1$)

- decision problem for property $\mathcal{P}$:
  - **input:** ANU R
  - **question:** $R \models \mathcal{P}$?

Theorem (Kari 94)

Any non-trivial property of limit sets of cellular automata is undecidable.

Limit sets

- $\mathcal{P}$ is **effectively non-trivial** if $\exists$ a poly(n) algo A(n) s.t.
  - $A(n) = (R_+, R_-)$
  - R_+ and R_- with n nodes
  - $R_+ \models \mathcal{P}$ and $R_- \not\models \mathcal{P}$

Limit sets

- $\mathcal{P}$ is **effectively non-trivial** if $\exists$ a poly(n) algo A(n) s.t.
  - $A(n) = (R_+, R_-)$
  - R_+ and R_- with n nodes
  - $R_+ \models \mathcal{P}$ and $R_- \not\models \mathcal{P}$

Theorem

Any effectively non-trivial property of limit sets of ANU is PSPACE-hard.

Limit sets

- $\mathcal{P}$ is **effectively non-trivial** if $\exists$ a poly(n) algo A(n) s.t.
  - $A(n) = (R_+, R_-)$
  - R_+ and R_- with n nodes
  - $R_+ \models \mathcal{P}$ and $R_- \not\models \mathcal{P}$

Theorem

Any effectively non-trivial property of limit sets of ANU is PSPACE-hard.

Question

A Rice-like theorem for properties of limit sets of DANU?

Limit sets

- $\mathcal{P}$ is **effectively non-trivial** if $\exists$ a poly(n) algo A(n) s.t.
  - $A(n) = (R_+, R_-)$
  - R_+ and R_- with n nodes
  - $R_+ \models \mathcal{P}$ and $R_- \not\models \mathcal{P}$

Theorem

Any effectively non-trivial property of limit sets of ANU is PSPACE-hard.

Question

A Rice-like theorem for properties of limit sets of DANU?

Theorem

Given a DANU F, knowing whether $|X - \Omega_F|$ is bounded by a (fixed) polynomial is Σ_3^p and co-NP-hard.

(D)AN as Succinct Graphs

- for any DAN F, $G_F = (X, E_F)$ is the dynamics graph:

$$(x, y) \in E_F \Leftrightarrow F(x) = y$$

- same for AN R

(D)AN as Succinct Graphs

- for any DAN F, $G_F = (X, E_F)$ is the dynamics graph:

$$(x, y) \in E_F \Leftrightarrow F(x) = y$$

- same for AN R

▷ *Boolean circuits for F = **succinct** description of G_F*

(D)AN as Succinct Graphs

- for any DAN F, $G_F = (X, E_F)$ is the dynamics graph:

$$(x, y) \in E_F \Leftrightarrow F(x) = y$$

- same for AN R

  ▷ *Boolean circuits for F = **succinct** description of G_F*

- black box / oracle: C circuit with two n-bit inputs:

$$(x, y) \in E_C \Leftrightarrow C(x, y) = 1$$

(D)AN as Succinct Graphs

- for any DAN F, $G_F = (X, E_F)$ is the dynamics graph:

$$(x, y) \in E_F \Leftrightarrow F(x) = y$$

- same for AN R

▷ *Boolean circuits for F = **succinct** description of G_F*

- black box / oracle: C circuit with two n-bit inputs:

$$(x, y) \in E_C \Leftrightarrow C(x, y) = 1$$

General question: *what can we decide effectively about graph G_C by opening and looking inside the black box C?*

FO logic for DAN

- quantification over configurations + relations $\rightarrow$ and $=$
- examples:
  - $\exists x : x \rightarrow x$
  - $\forall x, \exists y : y \rightarrow x$

FO logic for DAN

- quantification over configurations + relations $\rightarrow$ and $=$
- examples:
  - $\exists x : x \rightarrow x$
  - $\forall x, \exists y : y \rightarrow x$

- given FO formula ϕ, problem $\mathcal{P}_\phi$
  - **input:** a DAN F
  - **question:** does $G_F \models \phi$?
- say ϕ is ω-*nontrivial* if
  - $G_F \models \phi$ for infinitely many F
  - $G_F \not\models \phi$ for infinitely many F

FO logic for DAN

- quantification over configurations + relations $\to$ and $=$
- examples:
  - $\exists x : x \to x$
  - $\forall x, \exists y : y \to x$

- given FO formula ϕ, problem $\mathcal{P}_\phi$
  - **input:** a DAN F
  - **question:** does $G_F \models \phi$?
- say ϕ is ω-*nontrivial* if
  - $G_F \models \phi$ for infinitely many F
  - $G_F \not\models \phi$ for infinitely many F

Theorem

For any ω-nontrivial ϕ, $\mathcal{P}_\phi$ is either NP-hard or co-NP-hard.

Proof Ingredients

Pumping lemmas

For ϕ ω-nontrivial then $\exists$ graphs G_1, F, N s.t. $\forall n, k$

$$G_1(\otimes N)^n \models \phi \text{ and } G_1(\otimes N)^n \otimes F(\otimes N \text{ or F})^* \not\models \phi$$

or conversely with $\phi \leftrightarrow \neg\phi$

Proof Ingredients

Pumping lemmas

For ϕ ω-nontrivial then $\exists$ graphs G_1, F, N s.t. $\forall n, k$

$$G_1(\otimes N)^n \models \phi \text{ and } G_1(\otimes N)^n \otimes F(\otimes N \text{ or F})^* \not\models \phi$$

or conversely with $\phi \leftrightarrow \neg\phi$

- 4 lemmas: #components, cycle-length, degree, depth
- Ehrenfeucht-Fraïssé games + Hanf's local lemma

Proof Ingredients

Pumping lemmas

For ϕ ω-nontrivial then $\exists$ graphs G_1, F, N s.t. $\forall n, k$

$$G_1(\otimes N)^n \models \phi \text{ and } G_1(\otimes N)^n \otimes F(\otimes N \text{ or F})^* \not\models \phi$$

or conversely with $\phi \leftrightarrow \neg\phi$

- 4 lemmas: #components, cycle-length, degree, depth
- Ehrenfeucht-Fraïssé games + Hanf's local lemma

Question

Same theorem with uniform alphabet?

- $\forall x, \exists y : x \neq y \wedge x \to y \wedge y \to x$

MSO logic for AN

(work in progress)

- MSO = FO + quantification over sets of configurations + relation $x \in S$

$$\forall S, \forall S' : S \cap S' = \emptyset \wedge S \cup S' = X \Rightarrow$$
$$\exists x \in S, \exists y \in S', x \rightarrow y \vee y \rightarrow x$$

MSO logic for AN

(work in progress)

- MSO = FO + quantification over sets of configurations + relation $x \in S$

$$\forall S, \forall S' : S \cap S' = \emptyset \wedge S \cup S' = X \Rightarrow$$
$$\exists x \in S, \exists y \in S', x \to y \vee y \to x$$

- given MSO formula ψ, problem $\mathcal{P}_\psi$
  - **input:** AN R
  - **question:** does $G_R \models \psi$?

MSO logic for AN

(work in progress)

- MSO = FO + quantification over sets of configurations + relation $x \in S$

$$\forall S, \forall S' : S \cap S' = \emptyset \wedge S \cup S' = X \Rightarrow \exists x \in S, \exists y \in S', x \to y \vee y \to x$$

- given MSO formula ψ, problem $\mathcal{P}_\psi$
  - **input:** AN R
  - **question:** does $G_R \models \psi$?
- ψ is ω-nontrivial for *bounded-treewidth* if there is k with
  - $G_R \models \phi$ for infinitely many R with G_R of treewidth $\leq k$
  - $G_R \not\models \phi$ for infinitely many R with G_R of treewidth $\leq k$

MSO logic for AN

(work in progress)

- MSO = FO + quantification over sets of configurations + relation $x \in S$

$$\forall S, \forall S' : S \cap S' = \emptyset \wedge S \cup S' = X \Rightarrow \exists x \in S, \exists y \in S', x \to y \vee y \to x$$

- given MSO formula ψ, problem $\mathcal{P}_\psi$
  - **input:** AN R
  - **question:** does $G_R \models \psi$?
- ψ is ω-nontrivial for *bounded-treewidth* if there is k with
  - $G_R \models \phi$ for infinitely many R with G_R of treewidth $\leq k$
  - $G_R \not\models \phi$ for infinitely many R with G_R of treewidth $\leq k$

Theorem

Let ψ be ω-nontrivial for bounded treewidth then $\mathcal{P}_\psi$ is either NP-hard or co-NP-hard.

Proof ingredients

(work in progress)

Lemma (saturating graph)

$\forall \psi, \exists G_\psi : \forall G, G \sqcup G_\psi \models \psi$ **or** $\forall G, G \sqcup G_\psi \not\models \psi$

Proof ingredients

(work in progress)

Lemma (saturating graph)

$\forall\psi, \exists G_\psi : \forall G, G \sqcup G_\psi \models \psi$ **or** $\forall G, G \sqcup G_\psi \not\models \psi$

Pumping lemma

If ψ is ω-nontrivial for bounded treewidth then $\exists G_1, G_2, G_3$ s.t. $\forall n$, $G_1 \left(\oplus G_2\right)^n \oplus G_3 \models \psi$.

Proof ingredients

(work in progress)

Lemma (saturating graph)

$\forall \psi, \exists G_\psi : \forall G, G \sqcup G_\psi \models \psi$ **or** $\forall G, G \sqcup G_\psi \not\models \psi$

Pumping lemma

If ψ is ω-nontrivial for bounded treewidth then $\exists G_1, G_2, G_3$ s.t. $\forall n$, $G_1 \left(\oplus G_2\right)^n \oplus G_3 \models \psi$.

- Ehrenfeucht-Fraïssé games for MSO
- $G \sim G'$ iff $\forall H, G \oplus H \models \psi \Leftrightarrow G' \oplus H \models \psi$

Proof ingredients

(work in progress)

Lemma (saturating graph)

$\forall\psi, \exists G_\psi : \forall G, G \sqcup G_\psi \models \psi$ **or** $\forall G, G \sqcup G_\psi \not\models \psi$

Pumping lemma

If ψ is ω-nontrivial for bounded treewidth then $\exists G_1, G_2, G_3$ s.t. $\forall n$, $G_1 (\oplus G_2)^n \oplus G_3 \models \psi$.

- Ehrenfeucht-Fraïssé games for MSO
- $G \sim G'$ iff $\forall H, G \oplus H \models \psi \Leftrightarrow G' \oplus H \models \psi$

Question 1

Same theorem for DAN? for ANU? for DANU?

Proof ingredients

(work in progress)

Lemma (saturating graph)

$\forall\psi, \exists G_\psi : \forall G, G \sqcup G_\psi \models \psi$ **or** $\forall G, G \sqcup G_\psi \not\models \psi$

Pumping lemma

If ψ is ω-nontrivial for bounded treewidth then $\exists G_1, G_2, G_3$ s.t. $\forall n$, $G_1 \left(\oplus G_2\right)^n \oplus G_3 \models \psi$.

- Ehrenfeucht-Fraïssé games for MSO
- $G \sim G'$ iff $\forall H, G \oplus H \models \psi \Leftrightarrow G' \oplus H \models \psi$

Question 1

Same theorem for DAN? for ANU? for DANU?

Question 2

What about MSO formulas with unbounded treewidth models?

Obstacles for general MSO

(work in progress)

- $\mathrm{spec}(\psi) = \{|G| : G \models \psi\}$
- previous pumping lemmas: $\mathrm{spec}(\psi)$ contains $A_\psi\mathbb{N} + B_\psi$

Obstacles for general MSO

(work in progress)

- $\mathrm{spec}(\psi) = \{|G| : G \models \psi\}$
- previous pumping lemmas: $\mathrm{spec}(\psi)$ contains $A_\psi \mathbb{N} + B_\psi$
- $h : \mathbb{N} \to \mathbb{N}$ *time-constructible* if some Turing machine M halts in $h(n)$ steps on input n

Lemma

$\forall$ time-constructible h, $\exists\ \psi$ with $\mathrm{spec}(\psi) = \{h(n)^2\}$

Obstacles for general MSO

(work in progress)

- $\mathrm{spec}(\psi) = \{|G| : G \models \psi\}$
- previous pumping lemmas: $\mathrm{spec}(\psi)$ contains $A_\psi \mathbb{N} + B_\psi$
- $h : \mathbb{N} \to \mathbb{N}$ *time-constructible* if some Turing machine M halts in $h(n)$ steps on input n

Lemma

$\forall$ time-constructible h, $\exists\ \psi$ with $\mathrm{spec}(\psi) = \{h(n)^2\}$

Theorem (anti-pumping)

$\exists\ \psi$ s.t. $\forall$ primitive recursive π, $Im(\pi) \not\subseteq \mathrm{spec}(\psi)$

Obstacles for general MSO

(work in progress)

- $\mathrm{spec}(\psi) = \{|G| : G \models \psi\}$
- previous pumping lemmas: $\mathrm{spec}(\psi)$ contains $A_\psi\mathbb{N} + B_\psi$
- $h : \mathbb{N} \to \mathbb{N}$ *time-constructible* if some Turing machine M halts in $h(n)$ steps on input n

Lemma

$\forall$ time-constructible h, $\exists\ \psi$ with $\mathrm{spec}(\psi) = \{h(n)^2\}$

Theorem (anti-pumping)

$\exists\ \psi$ s.t. $\forall$ primitive recursive π, $Im(\pi) \not\subseteq \mathrm{spec}(\psi)$

Open

$\exists\ \psi$ such that $\mathcal{P}_\psi$ is neither P, nor NP-hard, nor co-NP-hard?

¡Gracias!