Parametrized Complexity of Freezing Dynamics

Journées SDA2 2020

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Freezing Dynamics

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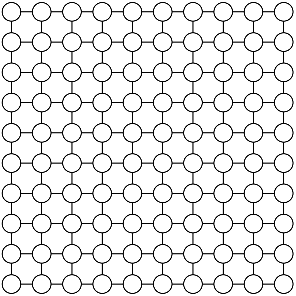

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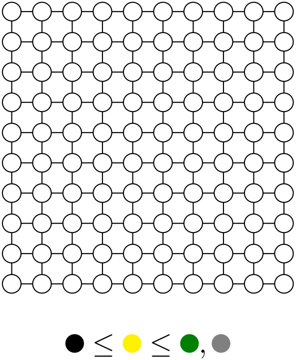

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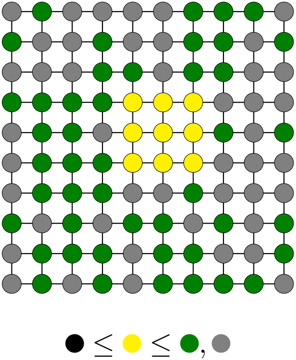

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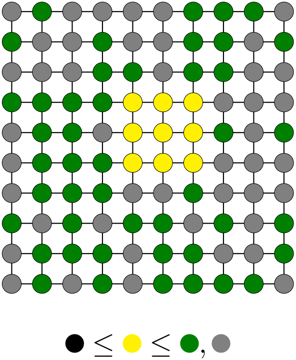

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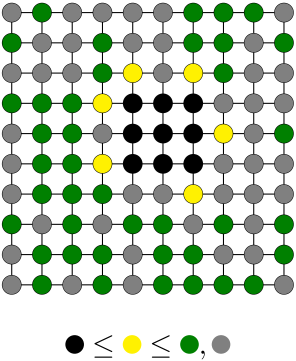

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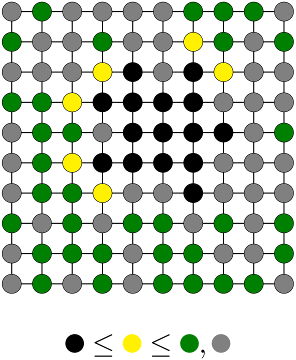

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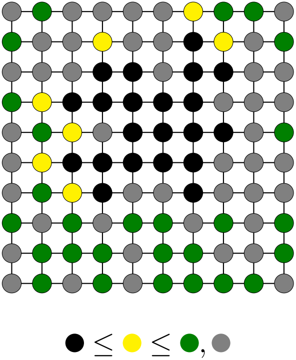

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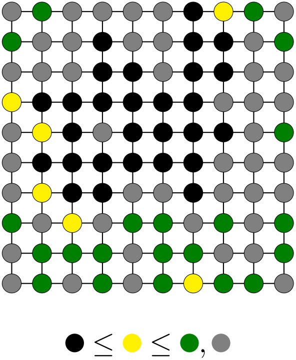

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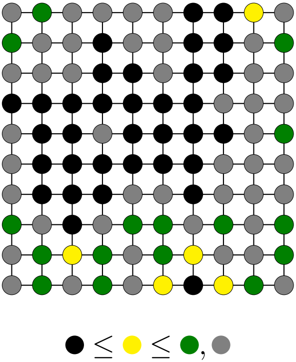

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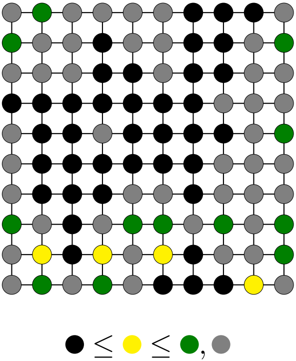

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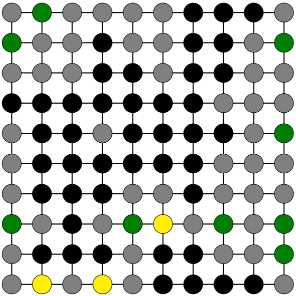

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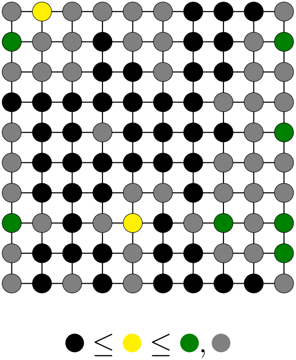

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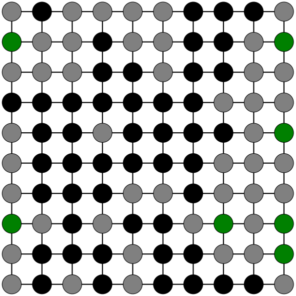

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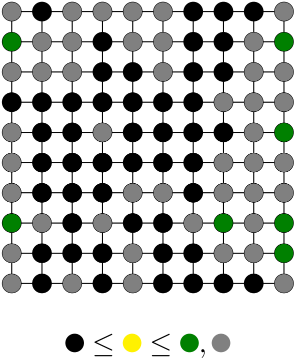

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- possibly **non-determinism**:

$$f_v : Q^{N(v)} \to 2^Q$$

Problems

- **prediction**(F, x, t, v)

$$F^t(x)_v = ?$$

- **nilpotency**(F)

$$\exists t : \; F^t \text{ is a constant map?}$$

- **predecessor**(F, t, y)

$$\exists t, x : \; F^t(x) = y?$$

- **density prediction**(F, t, x, q)

$$\frac{\#\{v \in V : F^t(x)_v = q\}}{\#V} \geq 50\%?$$

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FPT	**XP**

- FPT **reduction**: $P_1 \leq_{FPT} P_2$
  - $\phi(x, k)$ computable in time $\beta(k) \cdot |x|^c$
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- W-hierarchy inside XP (*W* stands for *weft*)

$$\text{FPT} \leq_{FPT} W[1] \leq_{FPT} W[2] \leq_{FPT} \cdots$$

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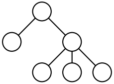
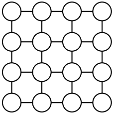
$$\text{VERTEX COVER} \leq_{FPT} \text{CLIQUE} \leq_{FPT} \text{DOMINATING SET}$$

Graph Theory





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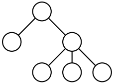
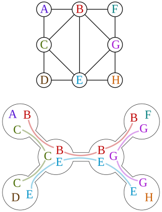
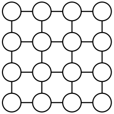
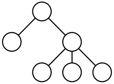


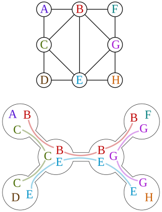


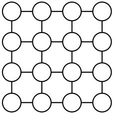


- treewidth

Graph Theory







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Courcelle's Theorem

Any MSO property of graphs is FPT for k =treewidth

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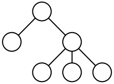

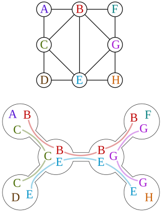

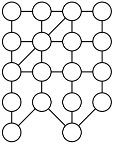

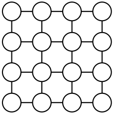

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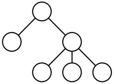

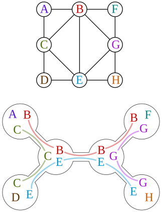

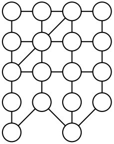

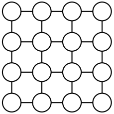

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Theorem (Robertson-Seymour '86 — Chekuri-Chuzhoy '16)

$gm(G) \leq tw(G) \leq poly(gm(G))$

Parametrized freezing dynamics

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- 3 results:
  1. efficient algorithm when **all** parameters are fixed
  2. $W[2]$-hardness for parameter $tw(G)$ or for Q
  3. uniform hardness results for graphs with $|V| \in poly(tw(G))$

Efficient algorithm

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- if single orbit, we can reconstruct the complete orbit in NC
- all problems presented before reduce to SPEC

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- **set defined** freezing rule $\phi : Q \times 2^Q \to Q$

$$f_v(x \in Q^{N(v)}) = \phi\big(x_v, \{x_u : u \in \mathbb{N}(v)\}\big)$$

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Theorem

There is a **set defined** freezing rule ϕ such that PREDECESSOR is NP-hard for ϕ on any dangerous family

- can't find a grid minor in polynomial time!
- routing lemma based on $L(1,1)$-coloring and perfect brambles (Kreutzer-Tazari '10)

Future work

- stronger hardness results in W-hierarchy
- FO^* hard on trees?
- hardness results under SETH for smaller treewidth growth
- algorithms for other bounded graph parameters
- infinite graphs!

$W[2]$-hardness

- degree kills our algorithm: very different from MSO!
- SPEC(tw): fix degree and Q, unique parameter = treewidth

Theorem

DOMINATING SET $\leq_{FPT}$ SPEC(tw)

- REGSPEC(Q):
  1. fix degree and treewidth, unique parameter = Q
  2. specify traces via regular expression

Theorem

DOMINATING SET $\leq_{FPT}$ REGSPEC(Q)