On simulation in automata networks

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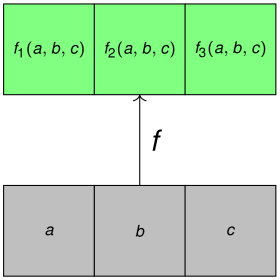
This talk is about **automata network**:

- computational model to study such problems
- modeling tool, synchronism vs. asynchronism in nature

General framework

- $f : Q^n \to Q^n$



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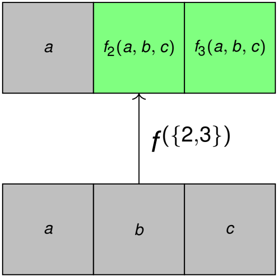

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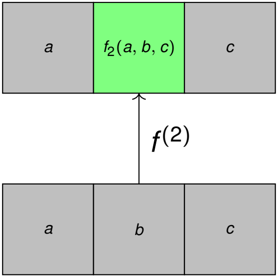

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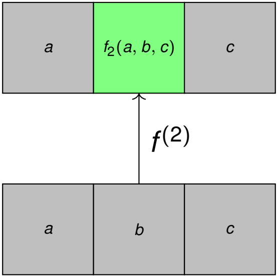

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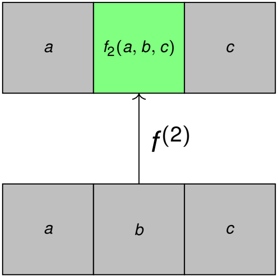

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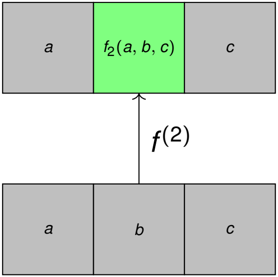

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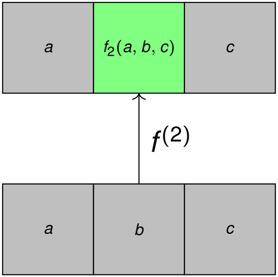

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Problems

How rich can be $\langle f \rangle_{\mathrm{Seq}}$? and $\langle f \rangle_{\mathrm{Asy}}$?
Given g, is there f with $g \in \langle f \rangle_{\mathrm{Seq}}$?

Known results

- $Q = \{0, \ldots, q-1\}$, $n, q \geq 2$
- $F(n, q)$: maps $Q^n \to Q^n$
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Key pattern in theorem: x,y,z with

- $g(x) = g(y)$
- $g^{-1}(z) = \emptyset$
- $x_i \neq y_i \Rightarrow z_i \notin \{x_i, y_i\}$

Fo... ...hi... ...e not sequentiali...

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Theorem (universality by initialization)

$\exists f \in F(n, q+1)$ such that $F(n,q) \sqsubseteq \langle f \rangle_{\mathrm{Seq}}$ (if $n \geq 3q$)

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by ini… g … b

- trick 1: encode q-configurations + 3 bits
- trick 2: synchronizing word

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Interaction graphs

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- interaction digraph $G(f) = (V, E)$
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- digraph D fixed:

$$F(D, q) = \{f \in F(n, q) : G(f) \subseteq D\}$$

- if D reflexive then

$$\langle F(D, q) \rangle_{\mathrm{Seq}} \subseteq \langle F(D, q) \rangle_{\mathrm{Asy}} = \langle F(D, q) \rangle$$

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Thank you!