Expansive Automata Networks

The Automata Factory 3

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Automata networks

- graph $G = (V, E)$
- alphabet $Q = \{0, \dots q-1\}$
- configurations $x \in Q^V$
- transition maps $f_v : Q^{N_v} \to Q$

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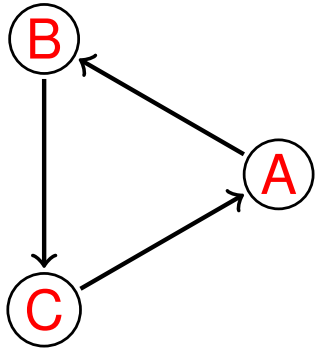
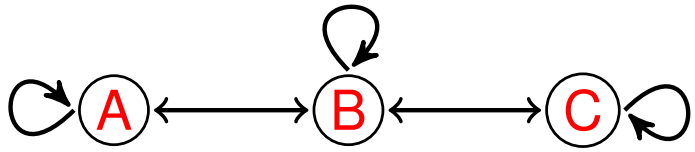
Example 1

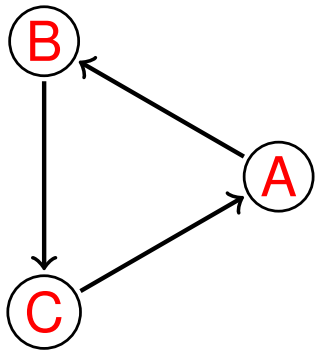


Example 2



with $Q = \{0, 1\}$ and $f_v(x) = \sum_{i \in N_v} x_i \bmod 2$

Examples



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t=1	0	1	0
t=2	0	0	1
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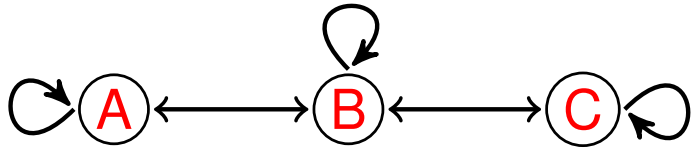

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$\rightsquigarrow$ **a 'topological dynamics' point of view**

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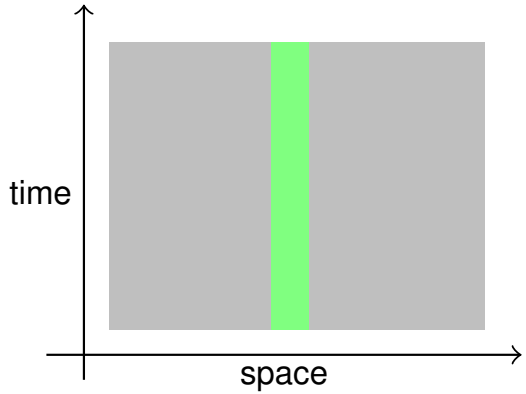

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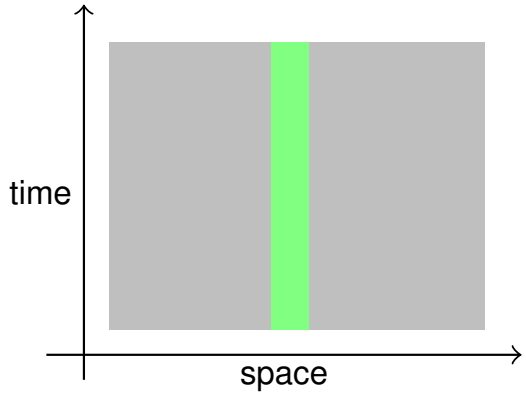

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- $\rightsquigarrow$ observability in control theory

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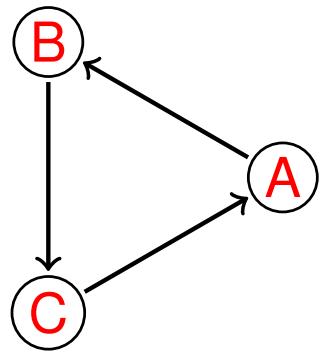
- expansive $\Rightarrow$ bijective
- example:



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2. $(\forall n)$ there is F with $T(F) = |Q|^n - |Q| - 1$

Exponential expansion time

Lexicographic order

1. 0 0 0
2. 0 0 1
3. 0 1 0
4. 0 1 1
5. 1 0 0
6. 1 0 1
7. 1 1 0
8. 1 1 1

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	Lexicographic order	**Twisted** lexicographic order
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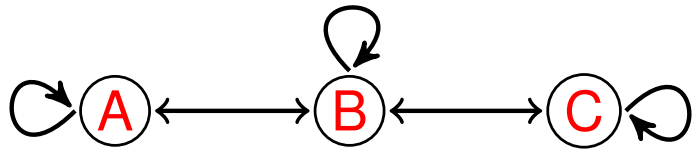
- **twisted** version is expansive
- T(**twisted**) = 5

Back to example 1



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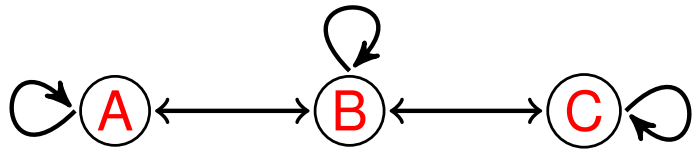

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  - $F(000) = 000$
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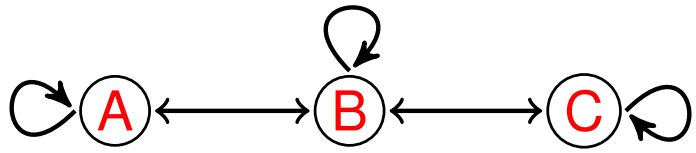

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- because of the update rules or because of the graph?
- something special for this kind of rules?

Linear networks

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- Then τ_v^n is a linear bijective map:

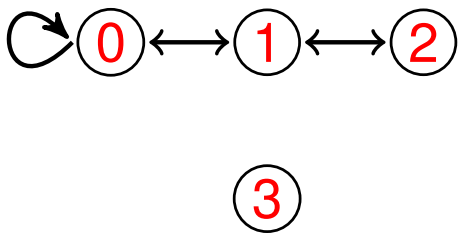
$$\tau_v^n = \begin{pmatrix} M_{1,v} & \cdots & M_{n,v} \\ \vdots & \cdots & \vdots \\ M_{1,v}^n & \cdots & M_{n,v}^n \end{pmatrix}$$

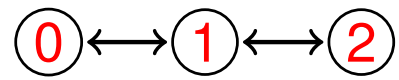
  where $F^t = (M_{i,j}^t)$
- for F linear:

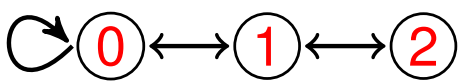
F expansive $\Leftrightarrow$ $det(\tau_v^n) \neq 0$ for all v

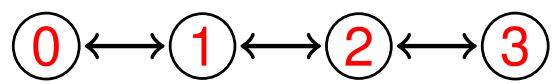
Graphs allowing expansiveness

- take any graph G
- can we define local rules f_v s.t. F is expansive?









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$P \in \mathbb{F}_q[X_1, \ldots, X_k]$, non-zero, total degree d, then:

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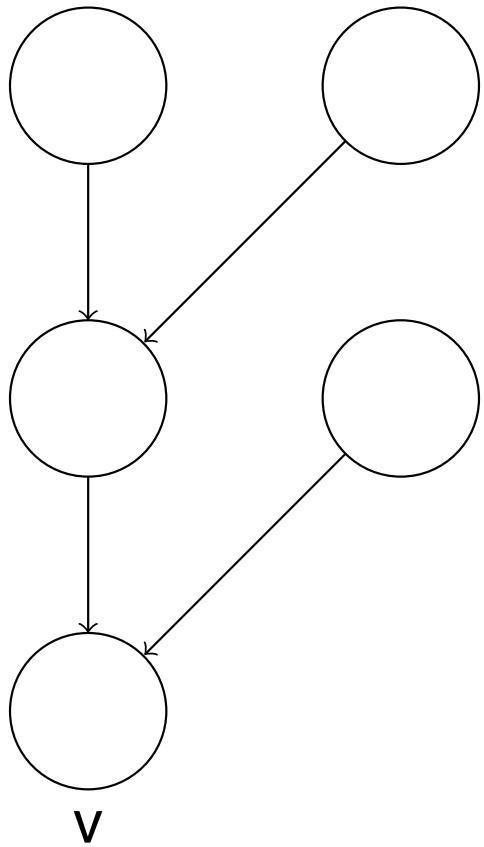
$$det(\tau_v^n) = \sum_{\sigma} (-1)^{s(\sigma)} \prod_{t=1}^{n} (\tau_v^n)_{\sigma(t),t}$$



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Open

Is (positive) expansivity decidable on cellular automata?

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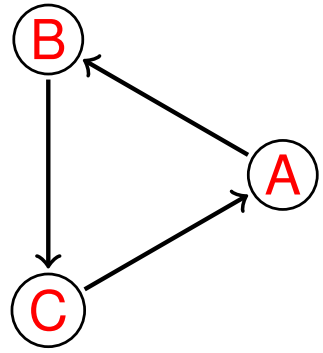
- all done by enumerating and counting
- **expansive counter** $\approx$ twisted lexicographic order

PSPACE-hardness construction

- layered construction

TEST
counter 1
counter 2
counter 3
x_1
x_2
x_3
instruction counter

TEST layer



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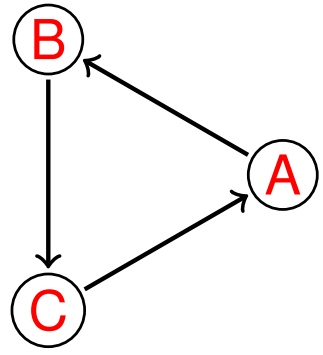

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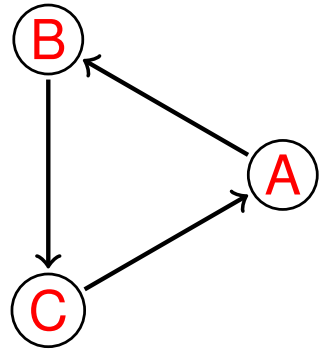

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- counter check trick: bad initialization $\Rightarrow$ rotate test layer
- Ψ true: rotate test layer
- Ψ false: do not rotate

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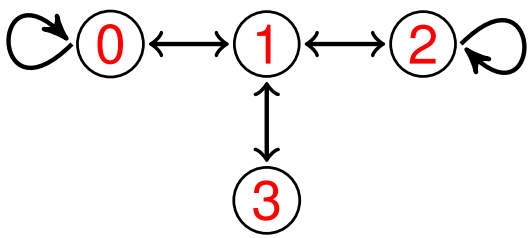

$q \equiv 2 \mod 4$

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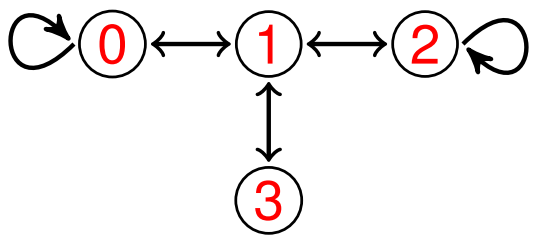
Theorem

For any Q there is an admissible G such that $E(Q, G) = \emptyset$.

- $n \sim q^{q^{q^2}}$ where $q = |Q|$
- q^2 is sufficient for **linear** networks

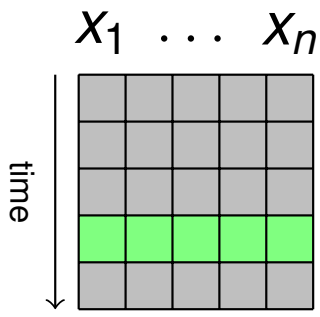


$q \equiv 2 \mod 4$

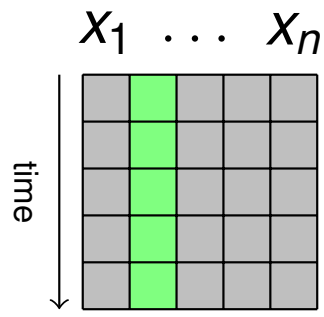
Open

Upper-bound on Q for all admissible graphs of fixed degree d?

Super-expansiveness



bijective



expansive in time n

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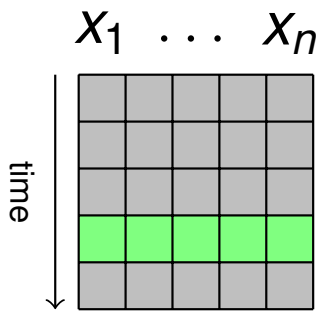

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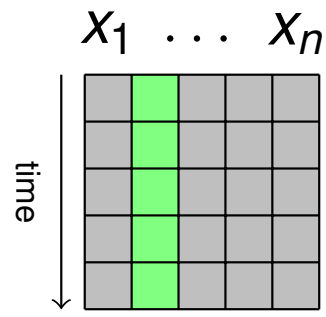

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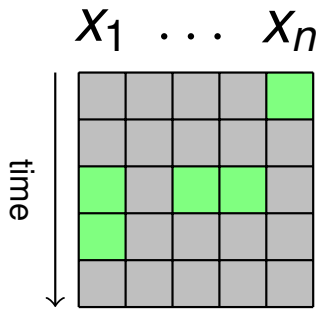

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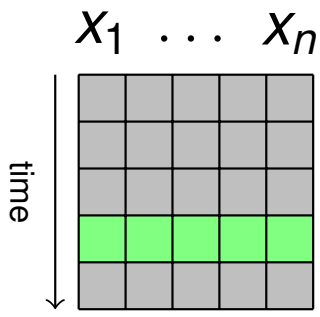

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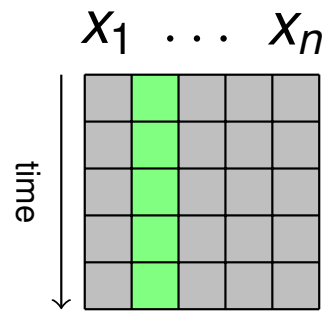

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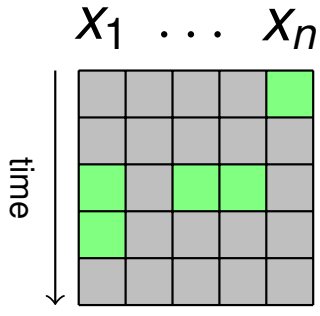

- $O = \big((v_1, t_1), \ldots, (v_n, t_n)\big)$
- $\tau_O = x \mapsto \big(F^{t_1}(x)_{v_1}, \ldots, F^{t_n}(x)_{v_n}\big)$

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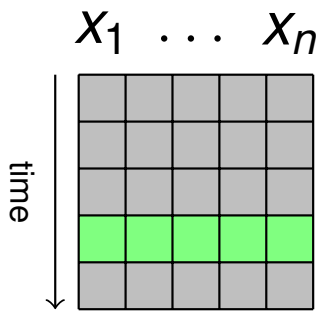

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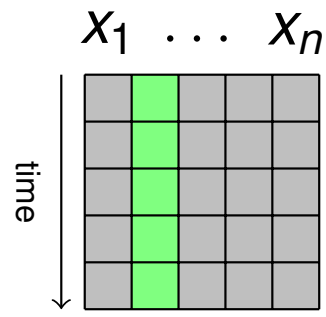
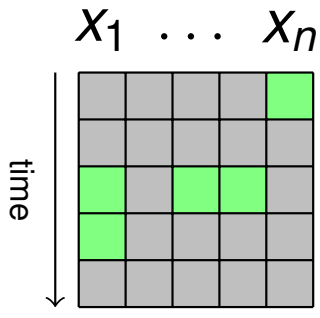


expansive in time n



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Definition

F super-expansive if τ_O injective for any O of length n.

Super-expansiveness

- F super-expansive on $G \Rightarrow$ G complete graph

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Theorem

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$$\left\{x_1 \cdots x_n F(x)_1 \cdots F(x)_n \cdots F^n(x)_1 \cdots F^n(x)_n : (x_i) \in Q^n\right\}$$

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- Singleton bound: $\#\text{words} \leq q^{\text{length}-\text{distance}+1}$
- In our case: $q^n \leq q^{n^2-(n^2-n+1)+1}$

Going further

- graph of bounded degree
- expansion frequency
- block-sequential update schedules
- link with other “topological” properties
- observability in general