Freezing Cellular Automata

Discrete Models of Complex Systems — Le STUDIUM

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Definitions

- **configurations:** $Q^{\mathbb{Z}^D}$, Q finite
- **topology:**

$$d(x, y) = 2^{-\min\{\|z\| \in \mathbb{Z}^D : x_z \neq y_z\}}$$

- **cellular automaton:** $F : Q^{\mathbb{Z}^D} \to Q^{\mathbb{Z}^D}$

$$\forall z \in \mathbb{Z}^D, F(c)_z = f\big(j \in V \mapsto c_{z+j}\big)$$

$V \subseteq \mathbb{Z}^D$ finite **neighborhood**, $f : Q^V \to Q$ **local rule**

Cold dynamics

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- F is **convergent** if $(F^t(x))_t$ is convergent for all x, *i.e.*

$$\forall x, \exists x_\infty \;:\; d\left(F^t(x), x_\infty\right) \to 0$$

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  - $\Sigma =$ your favorite SFT
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General questions

- how long does it take for a cell to freeze?
- how rich can be the transient?
- how complex can be $F^\omega(x)$ compared to x ?

Some facts in 1D

(with E. Goles and N. Ollinger)

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	Freezing	k-change	Convergent
Membership	PTIME	undecidable	undecidable
Turing-universal	YES	YES	YES
Prediction	LOGSPACE	LOGSPACE	P-complete
CC(Prediction)	$O(\log(n))$	$O(\log(n))$	can be $\geq \sqrt{n}$
Nilpotency	decidable	decidable	decidable

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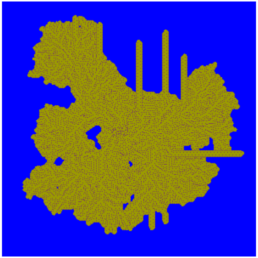
Carton-Guillon-Reiter, 2018

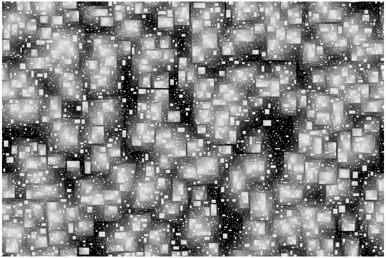
One-way freezing CA recognize the same languages as sumless/copyless counter machines.

2D freezing CA

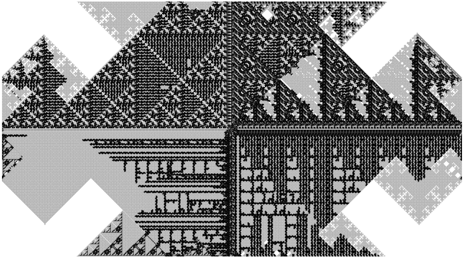
Life without death



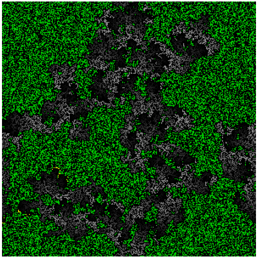
Bootstrap percolation



Random example



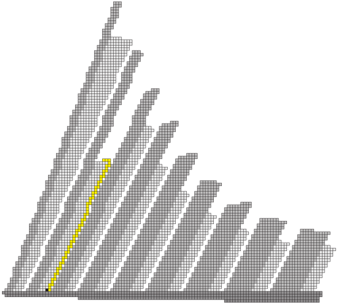
SIR propagation models



Uncomputable limit points

Theorem (Lathrop-Lutz-Patitz-Summers, 2010)

A directed self-assembly aTAM system can do this:



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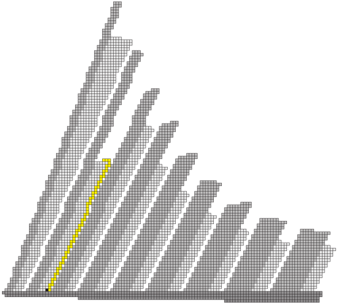

Corollary

$\exists$ F freezing and x finite such that $F^{\omega}(x)$ is uncomputable.

Simulations

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1 step	T steps

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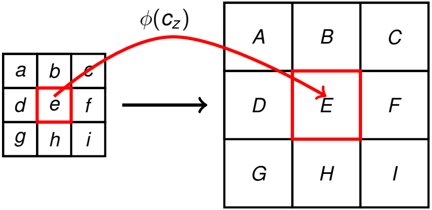
injective



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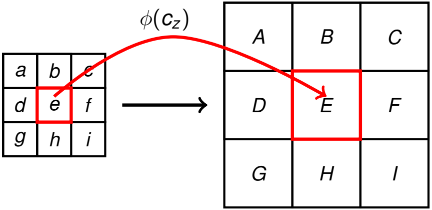
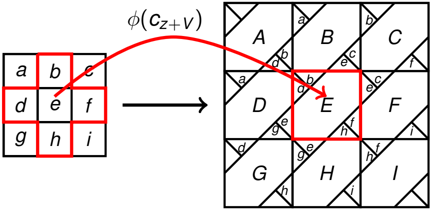


2. **context sensitive:**

   $\phi : Q_G^V \to Q_F^{m \times m}$
   injective global map



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Well-known

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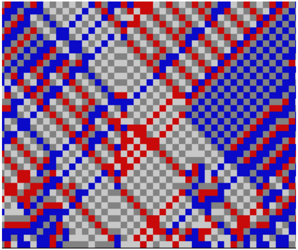

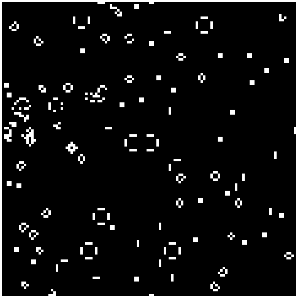

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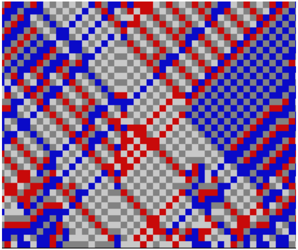

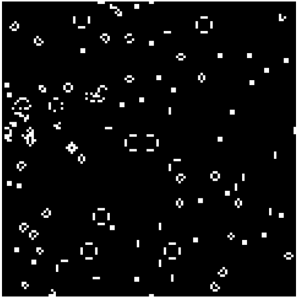

- *F* is **freezing-universal** if it is freezing and can simulate any freezing CA

Obstacles to Freezing-universality

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No freezing-universal CA in 1D, whatever the simulation mode.

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Theorem

No 1-change freezing universal in 2D with von Neumann neighborhood.

Intuition: Jordan's curves

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  - 1 change enough in 3D or 2D with Moore neighb.

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- information crossing:
  - 2 changes enough in 2D with von Neumann neighb.
  - 1 change enough in 3D or 2D with Moore neighb.
- synchronization ($\neq$ from aTAM universality)

5 states / von Neumann neighb.

- Freezing order: $\square, \blacksquare \geq {\color{blue}\blacksquare}, {\color{green}\blacksquare} \geq {\color{red}\blacksquare}$
- Transitions changing the state:

if $\square \in \{n, e, s, w\}$:

- $\begin{array}{ccc} & n & \\ w & \blacksquare & e \\ & s & \end{array} \mapsto {\color{red}\blacksquare}$ if ${\color{red}\blacksquare} \in \{n, e, s, w\}$
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if $\{n, e, w\} \subseteq \{\blacksquare, {\color{red}\blacksquare}\}$:

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Corollary

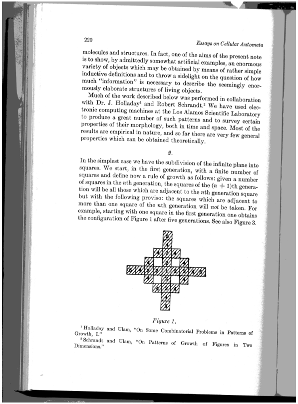
There is x eventually periodic such that $F^{\omega}(x)$ is uncomputable.

Rule 1

- $Q = \{0, 1\}$
- V von Neumann neighb.

$$F_1(x)_z = \begin{cases} 1 & \text{if } x_z = 1, \\ 1 & \text{if } \#_1(z + V) = 1, \\ 0 & \text{else.} \end{cases}$$

S. Ulam, 1960s



220 *Essays on Cellular Automata*

molecules and structures. In fact, one of the aims of the present note is to show, by admittedly somewhat artificial examples, an enormous variety of objects which may be obtained by means of rather simple inductive definitions and to throw a sidelight on the question of how much "information" is necessary to describe the seemingly enormously elaborate structures of living objects.

Much of the work described below was performed in collaboration with Dr. J. Holladay[1] and Robert Schrandt.[2] We have used electronic computing machines at the Los Alamos Scientific Laboratory to produce a great number of such patterns and to survey certain properties of their morphology, both in time and space. Most of the results are empirical in nature, and so far there are very few general properties which can be obtained theoretically.

2.

In the simplest case we have the subdivision of the infinite plane into squares. We start, in the first generation, with a finite number of squares and define now a rule of growth as follows: given a number of squares in the nth generation, the squares of the $(n + 1)$th generation will be all those which are adjacent to the nth generation square but with the following proviso: the squares which are adjacent to more than one square of the nth generation will *not* be taken. For example, starting with one square in the first generation one obtains the configuration of Figure 1 after five generations. See also Figure 3.

Figure 1.

[1] Holladay and Ulam, "On Some Combinatorial Problems in Patterns of Growth, I."

[2] Schrandt and Ulam, "On Patterns of Growth of Figures in Two Dimensions."

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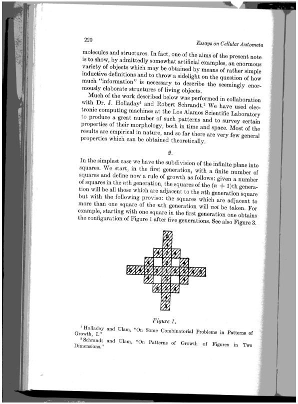

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- 1-change + von Neumann $\Rightarrow$ not freezing-universal

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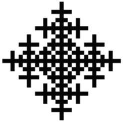

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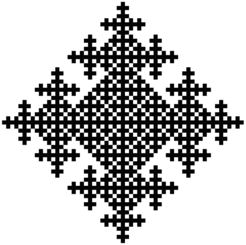

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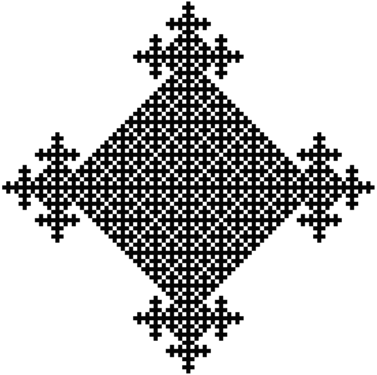

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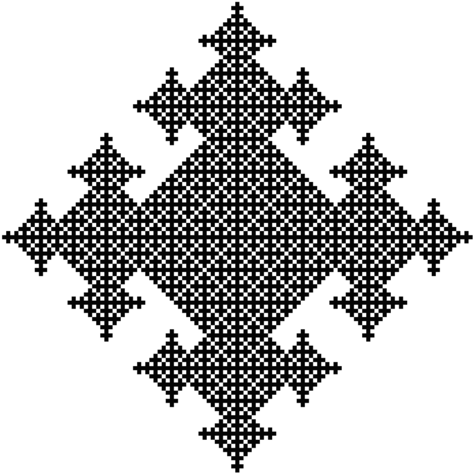

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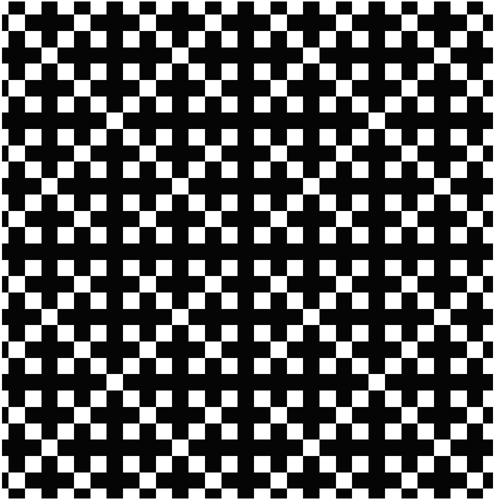

Fact

$F_1^\omega(1)_{(x,y)} = 1 \Leftrightarrow \nu_2(x) \neq \nu_2(y)$ with $\nu_2(a) = \max\{i : 2^i | a\}$

$F_1^\omega(x)$

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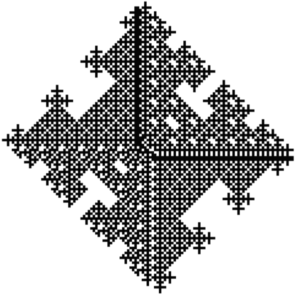

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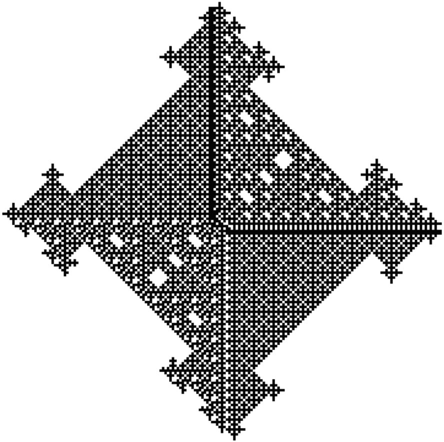

Any guess?

Rule 1 and 2-automaticity

(with E. Goles, P. Montealegre, V. Salo)

- $\overline{x}^k$: base-k representation of $x \in \mathbb{N}$
- $X \subseteq \mathbb{N}$ is k-automatic if $\{\overline{x}^k : x \in X\}$ is DFA-recognizable

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X is k-automatic **IFF** it is a column in the space-time diagram of a linear CA starting from an eventually periodic configuration.

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$$(9, 2) \mapsto \begin{matrix} 1\,0\,0\,1 \\ 0\,0\,1\,0 \end{matrix}$$

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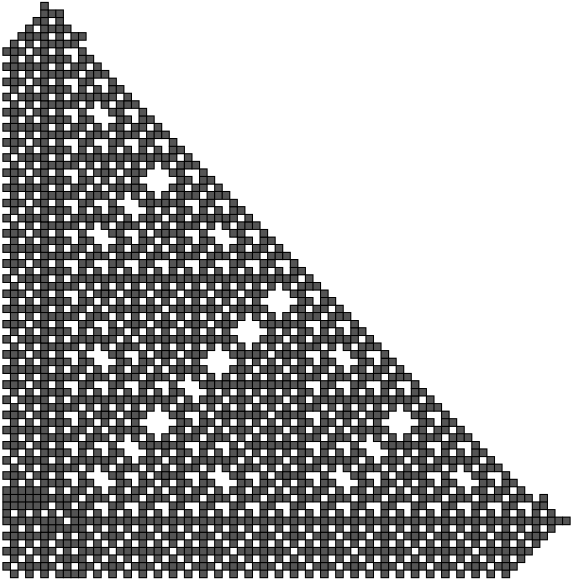
Claim

For any eventually periodic x, $F_1^\omega(x)$ is 2-automatic.

Idea of the proof



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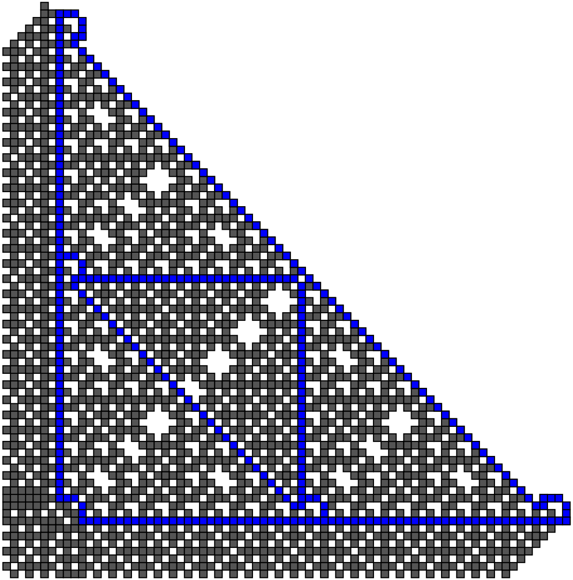

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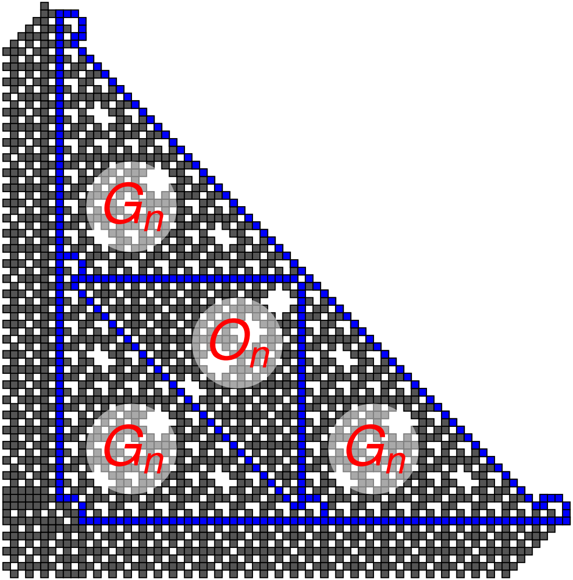

Questions

1. is “life without death” freezing-universal?
2. is there a bounded-change-universal CA?
3. is there a convergent-universal CA?
4. what freezing CA have 2-automatic limits starting from eventually periodic configurations?