Randomization in Cellular Automata

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Cellular Automata

- Q finite **alphabet**, configuration space $Q^{\mathbb{Z}}$

$$c = \cdots c_{-1} c_0 c_1 \cdots$$

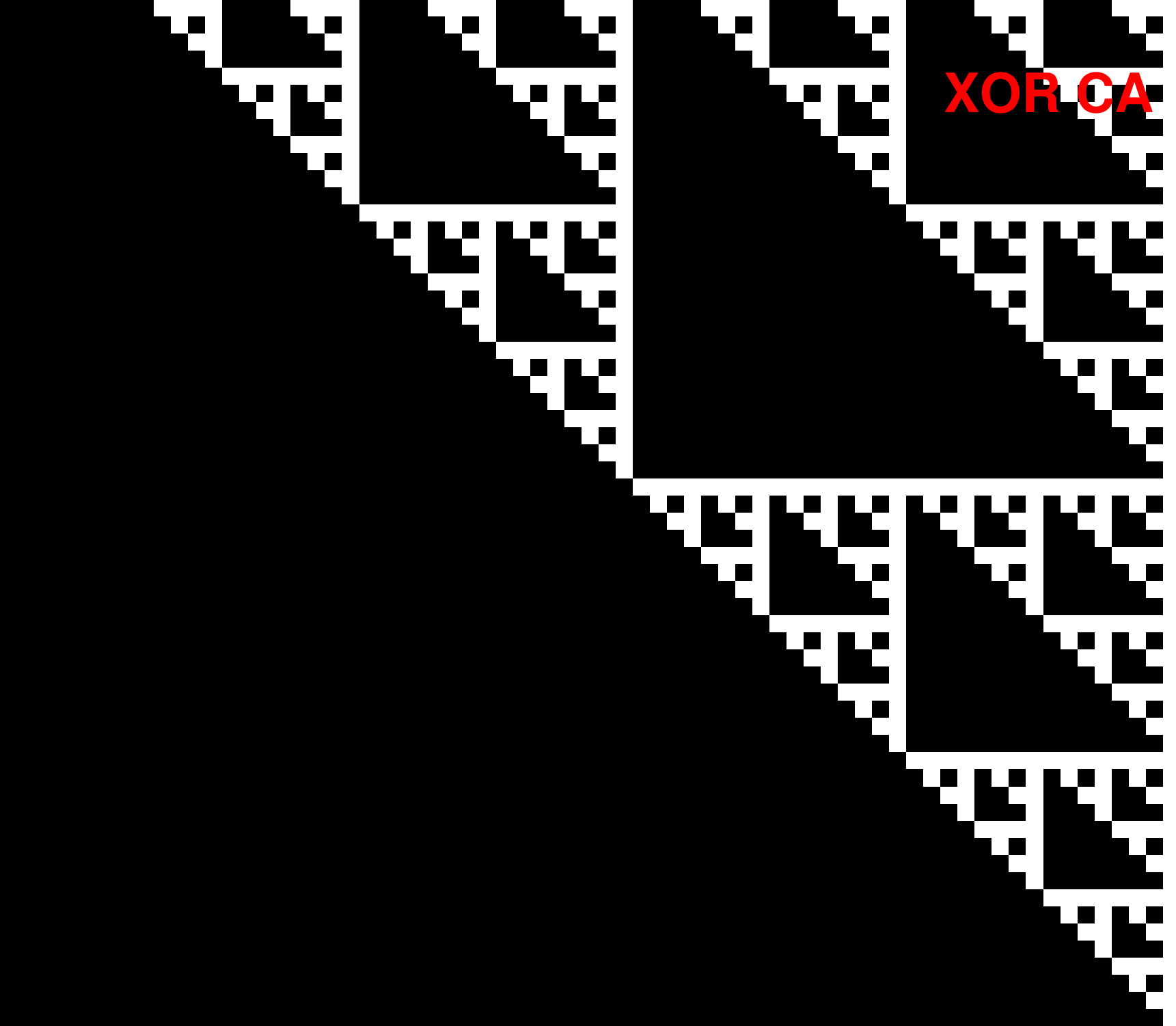
- **cellular automaton** $F : Q^{\mathbb{Z}} \to Q^{\mathbb{Z}}$

$$F(c)_z = f\left(j \in V \mapsto c_{z+j}\right)$$

$V \subseteq \mathbb{Z}$ finite **neighborhood**, $f : Q^V \to V$ **local rule**

- example with $Q = \{0, 1\}$: XOR CA

$$F(c)_z = c_z + c_{z+1} \bmod 2$$


XOR CA

CA and Probability Measures

- **cylinders** (base of open sets)

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- **NDB**: **N**on-**D**egenerate **B**ernoulli measure μ

$$\mu(u) = \prod \mu(u_i)\ ,\ \mu(q) > 0\ \forall q \in Q$$

- example: uniform Bernoulli measure, $\mu_0(u) = \#Q^{-|u|}$

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- iteration: $F\mu(u) = \mu\big(F^{-1}([u])\big)$

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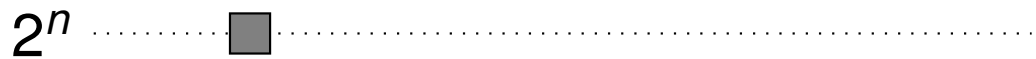

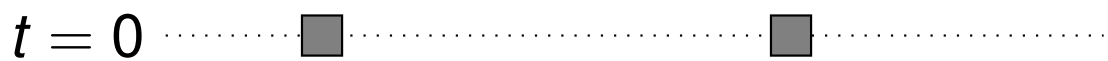

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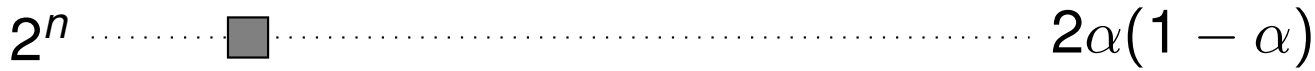

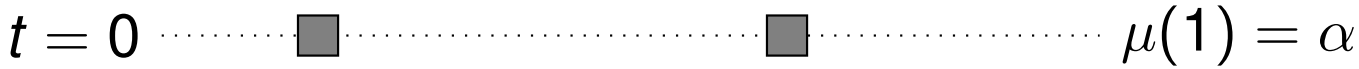

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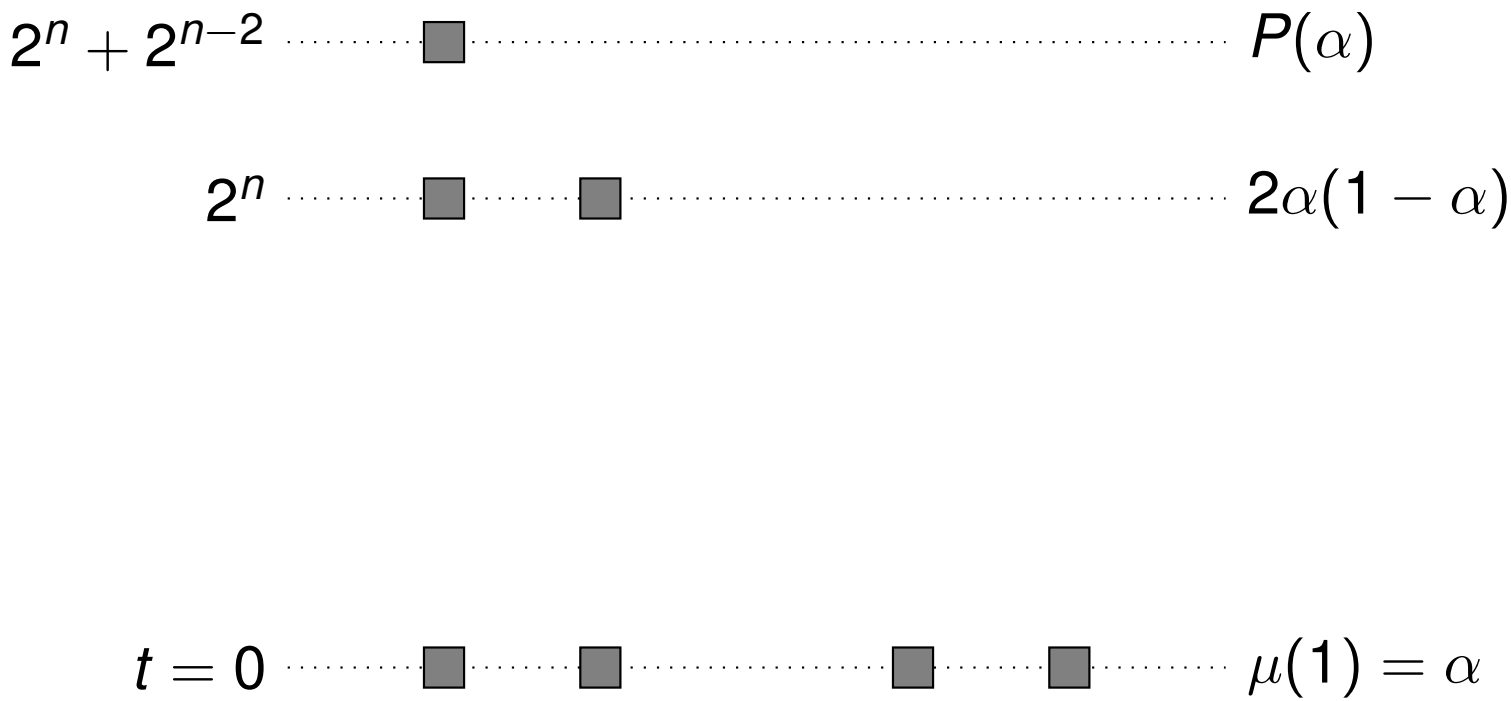

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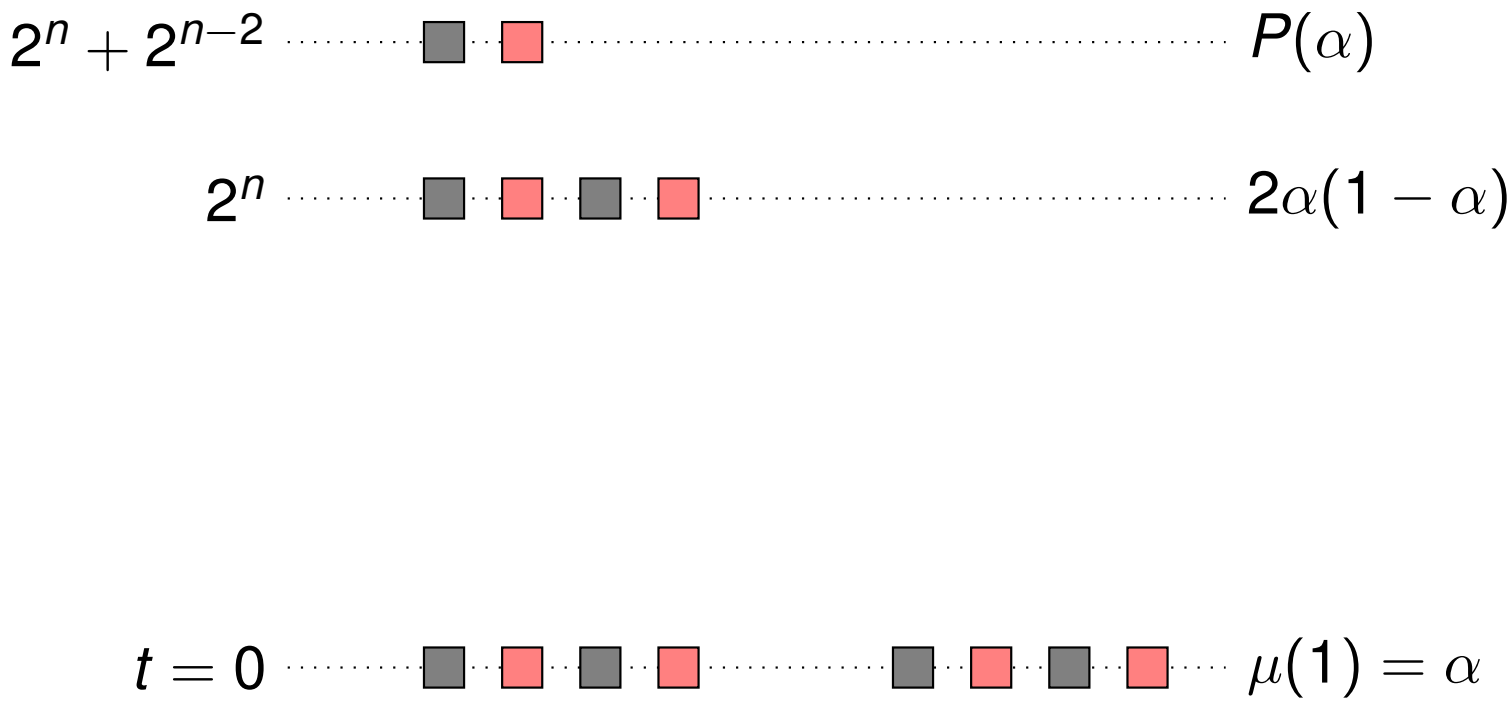

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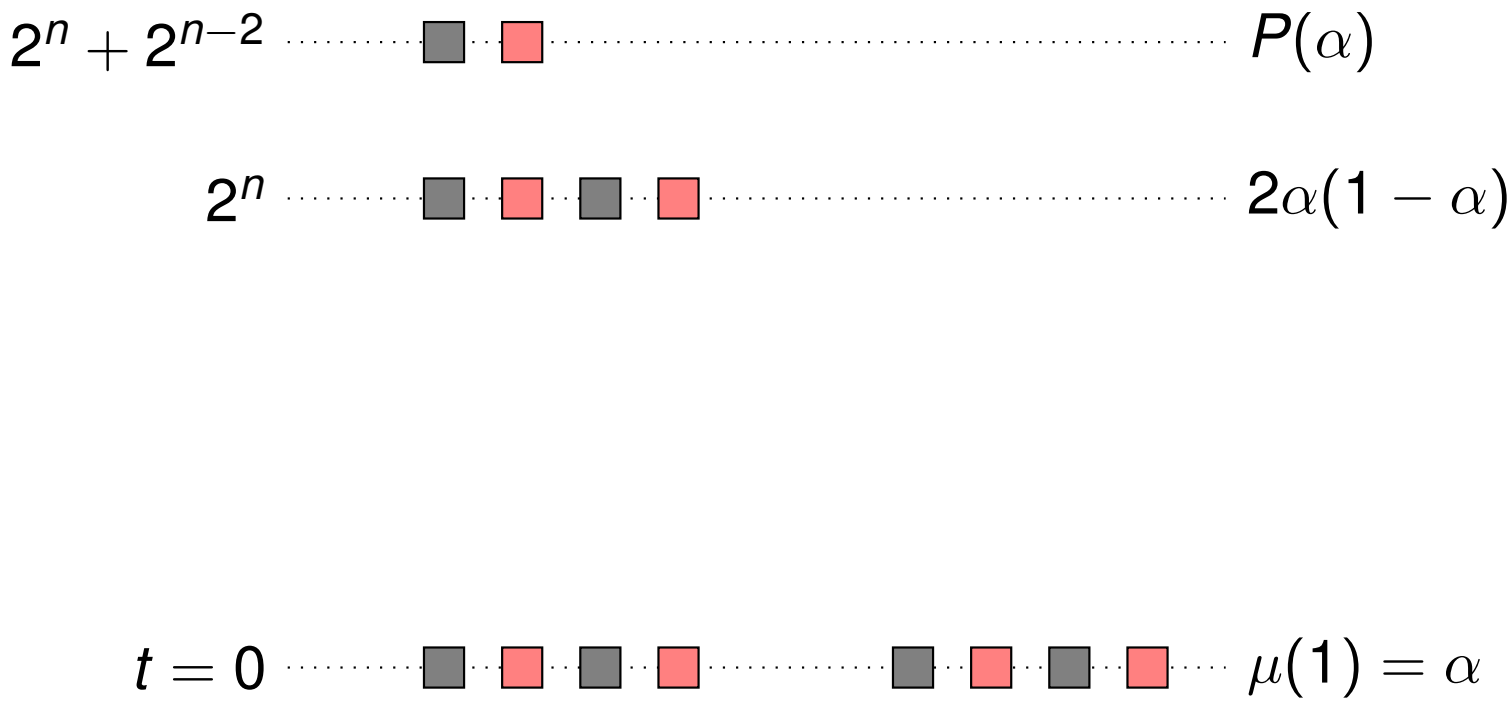

- $F^{t_n}\mu \to_n \mu_0$ for a good choice of t_n

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(t_n) subsequence of $\mathbb{N}$ of density 1

Some Results on Randomization

- **Miyamoto 1979, Lind 1984**

  XOR CA is randomizing for NDB measures.

- **Ferrari-Maass-Martinez-Ney 2000**

  $F(c)_z = \alpha c_z + \beta c_{z+1} \bmod p^l$, $\alpha, \beta \neq 0 \bmod p$
  is randomizing for measures with quick correlation decay.

- **Pivato-Yassawi 2002**

  $F(c)_z = \sum_{j \in J} \alpha_j c_{z+j}$ on Abelian group + conditions on α_j
  is randomizing for harmonically mixing measures.

- **Pivato-Yassawi 2004**

  $F(c)_z = \sum_{j \in J} \phi_j(c_{z+j})$ on Abelian group with $|J| \geq 2$ and ϕ_j commuting automorphisms
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This talk

- consider all **Abelian** CAs: $(Q, \oplus)$ Abelian group and

$$F(c \overline{\oplus} d) = F(c) \overline{\oplus} F(d)$$

  or equivalently

$$F(c)_z = \bigoplus_{i \in V} h_i(c_{z+i})$$

  with h_i endomorphisms of $(Q, \oplus)$

- **characterization** of randomization

- example of **strong** randomization

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- **glider:** *differences stay in a tube*
  $c \diamond d$ such that $\Delta(F^t(c), F^t(d)) \subseteq [\alpha t - \beta, \alpha t + \beta]$ for all t

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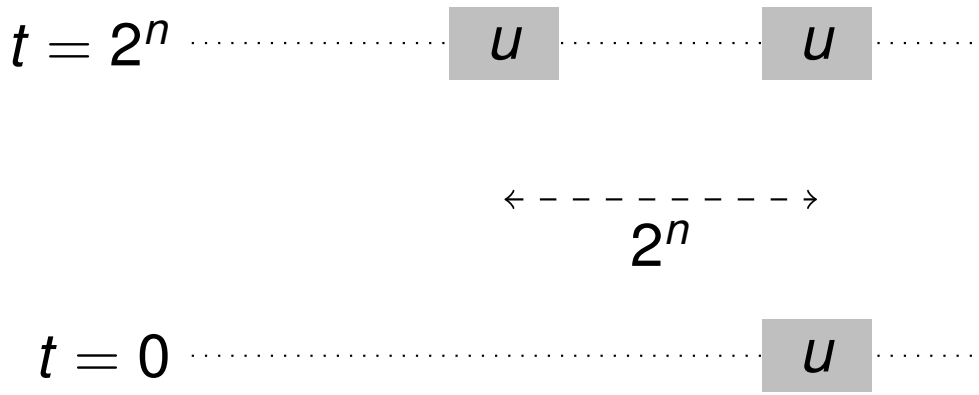

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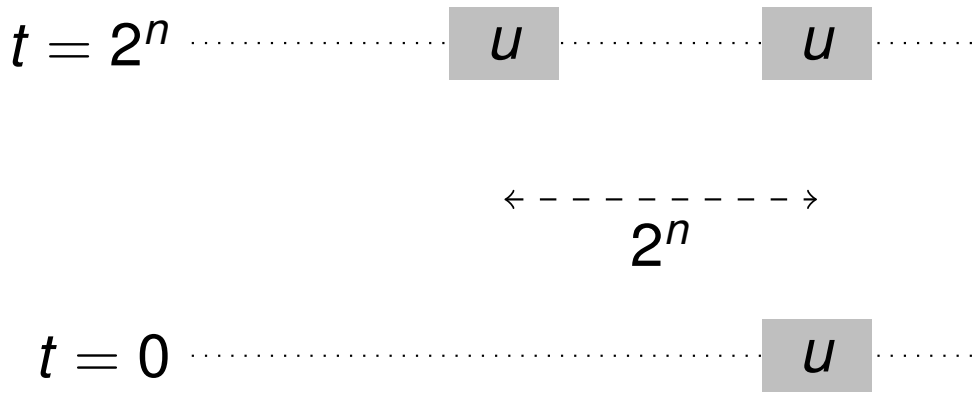

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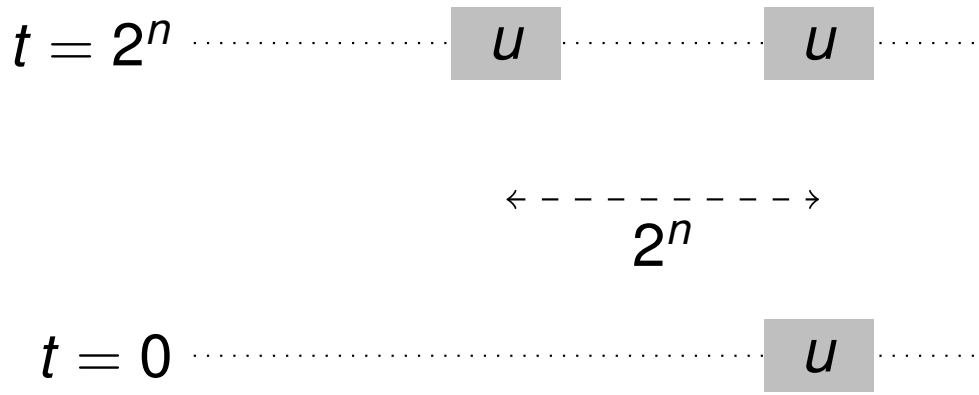
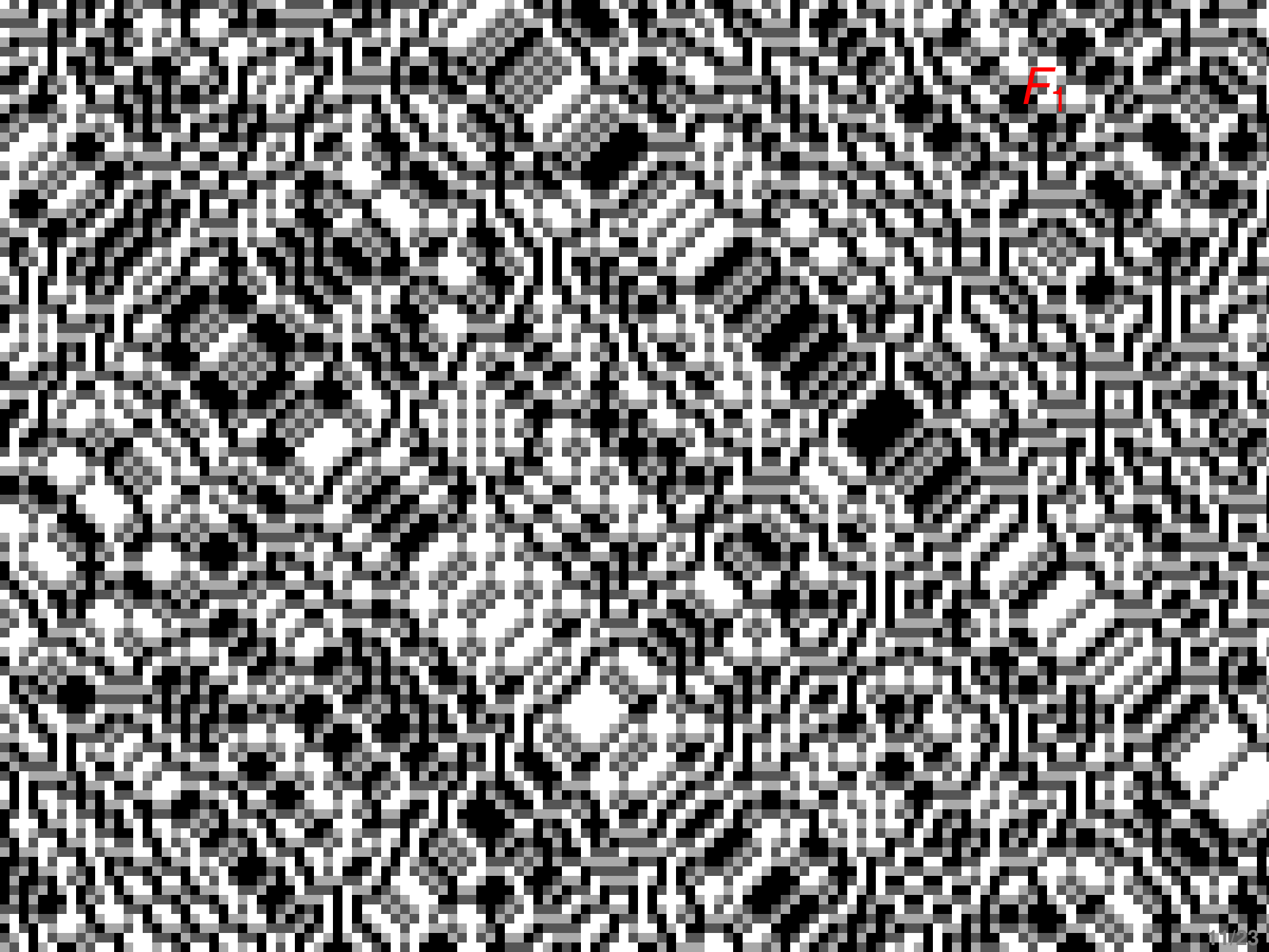
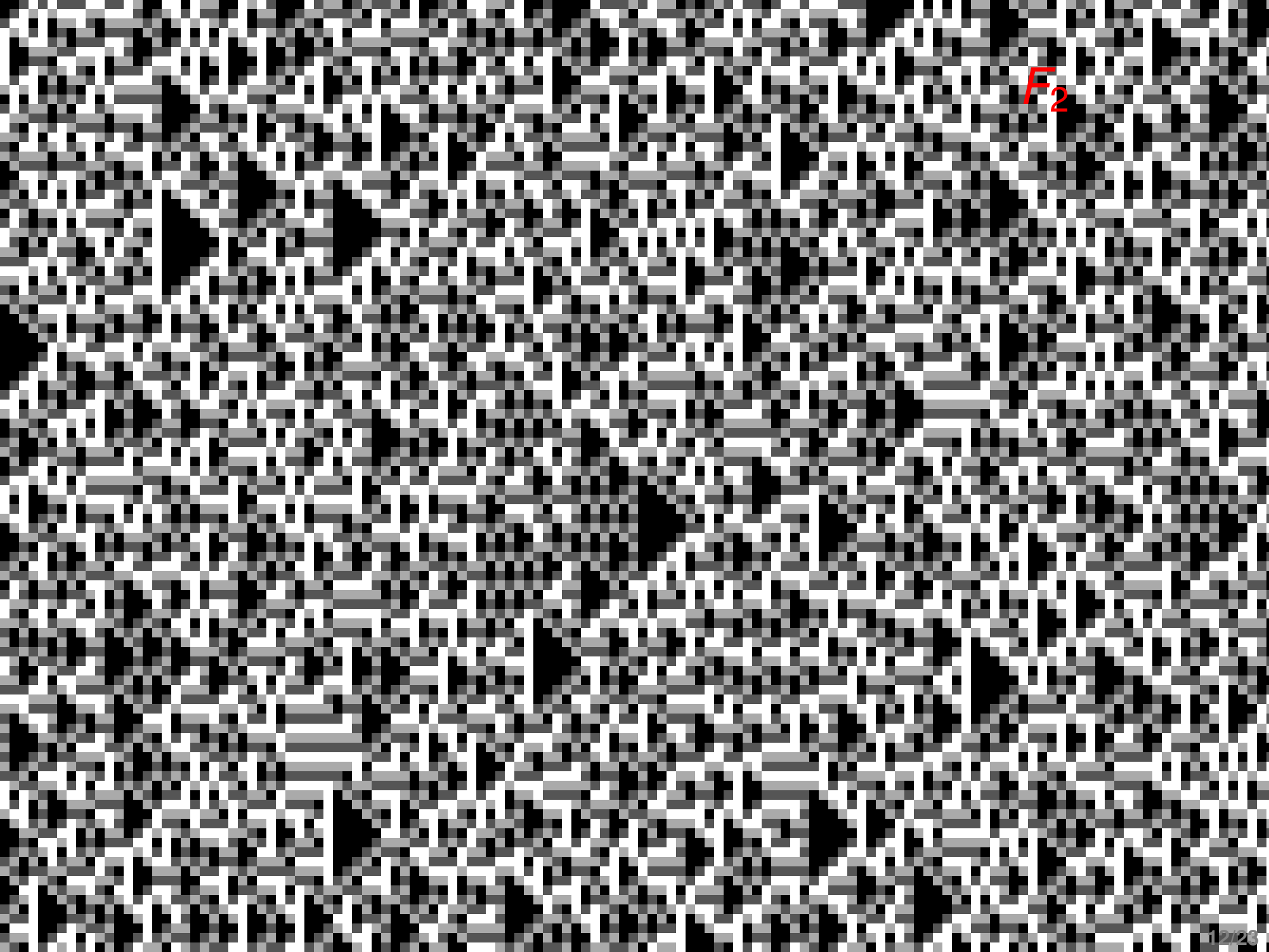
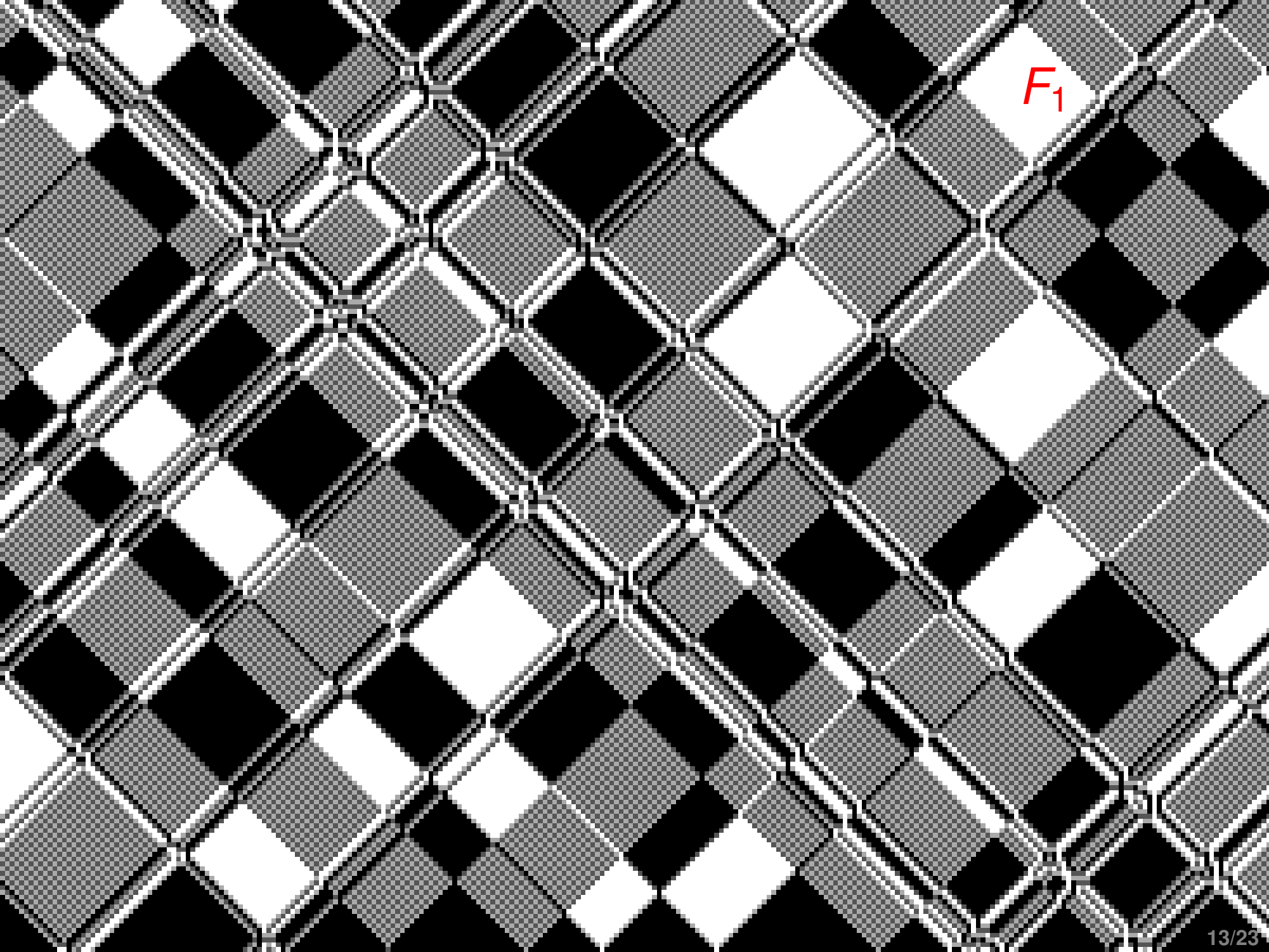


- XOR is strongly diffusive? **NO!** $F^{2^n}(c)_z = c_z \oplus c_{z+2^n}$
- examples with $(Q, \oplus) = \mathbb{Z}_2^2$
  1. $F_1(c)_z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot c_{z-1} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot c_z + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot c_{z+1}$
  2. $F_2(c)_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot c_z + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot c_{z+1}$


F_1


F_2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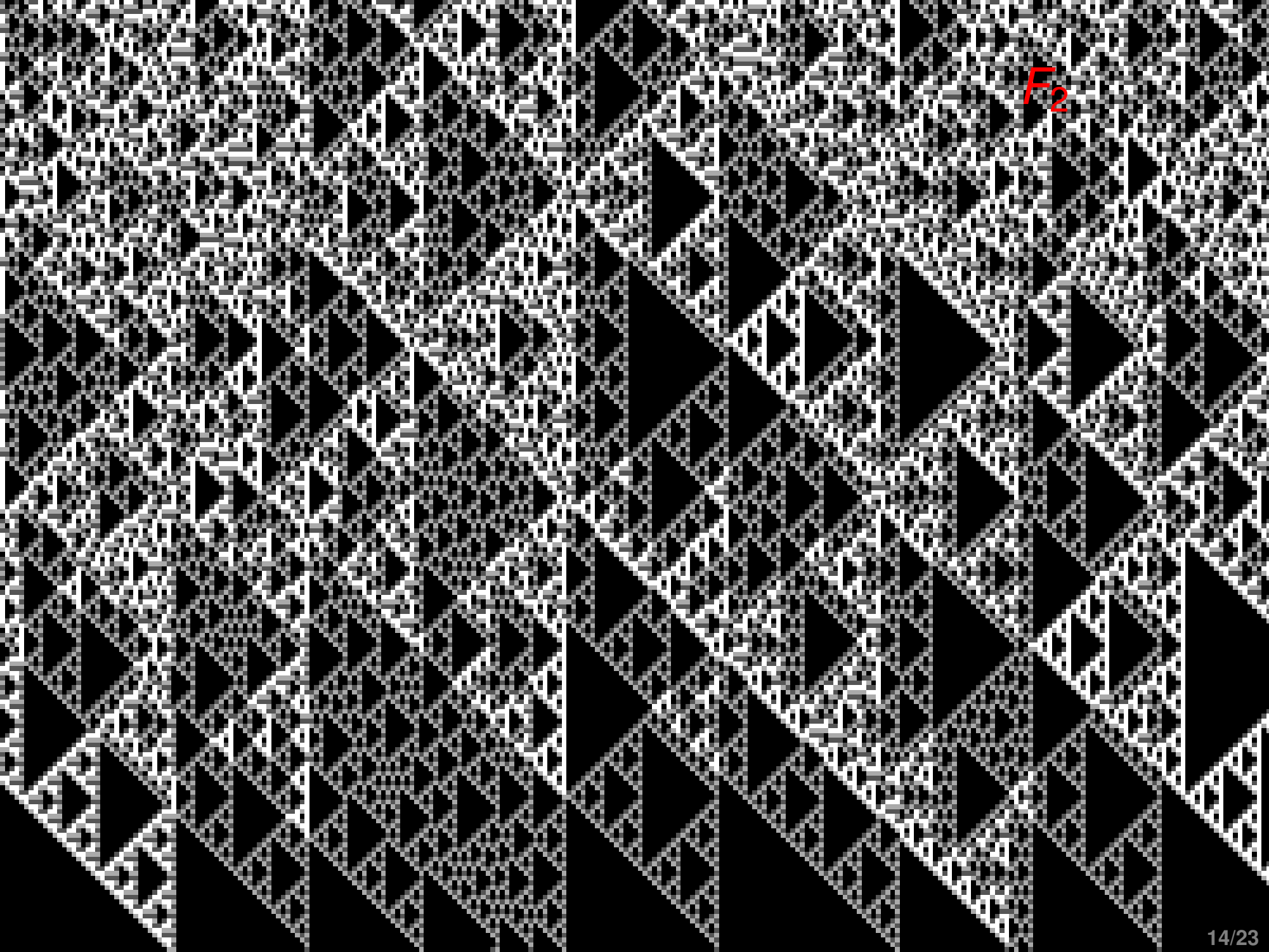
F_2

Main Theorem

Theorem

Let F be an Abelian CA, then the following are equivalent:

1. F has no glider
2. F is diffusive
3. F is randomizing for harmonically mixing measures
4. F is randomizing for NDB measures

Corollaries

For Abelian CA:

1. F and G randomizing $\Rightarrow$ $F \times G$ randomizing
2. F randomizing and reversible $\Rightarrow$ F^{-1} randomizing
3. F positively expansive $\Rightarrow$ F randomizing
4. F randomizing $\Rightarrow$ all its subautomata are randomizing
5. randomizing μ NDB $\Leftrightarrow$ randomizing μ harmonically mixing

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Question

Are the above statements still true for non-Abelian CA?

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2. harmonic analysis (inspired from Pivato-Yassawi)
   - characters: $\chi : Q^{\mathbb{Z}} \to \mathbb{C}$ continuous group morphism
   - Fourier coefficients: $\mu[\chi] = \int \chi d\mu$
   - $F\mu[\chi] = \mu[\chi \circ F]$
   - study $\chi \circ F^n$ to understand $F^n\mu$
   - harmonically mixing (HM) measures

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3. dual F^*: action of F on characters
   - F^* diffusive $\Leftrightarrow$ F randomizing
   - F^* has a glider $\Leftrightarrow$ F has a glider

Measures and Characters

- **character** of $Q^{\mathbb{Z}}$ = finite product of elementary characters

$$\chi(c) = \prod_z \chi_z(c_z) \text{ where } \chi_z : Q \to \mathbb{C} \text{ are almost all trivial}$$

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- for μ HM, $\mathrm{rank}(\chi \circ F^n) \to \infty \Rightarrow F^n\mu[\chi] \to 0$

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- $\chi_q = q' \mapsto \lambda^{<q,q'>}$ ($\lambda^p = 1$, $\lambda \neq 1$)
- $\hat{F}(c)_z = M_1^T \cdot c_{z-1} + M_0^T \cdot c_z + M_{-1}^T \cdot c_{z+1}$

Proof Tool: Dependencies

- δ_z^t: how cell z at time t depends on cell 0 at time 0

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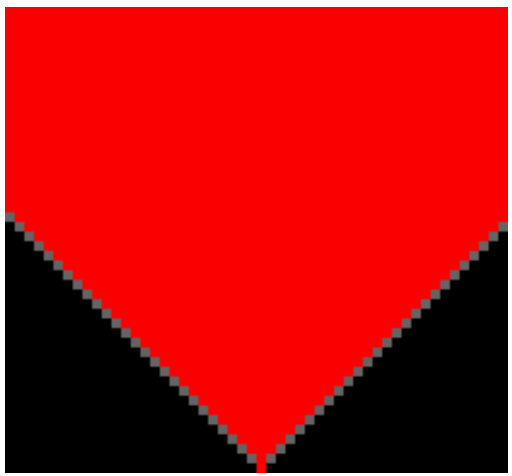
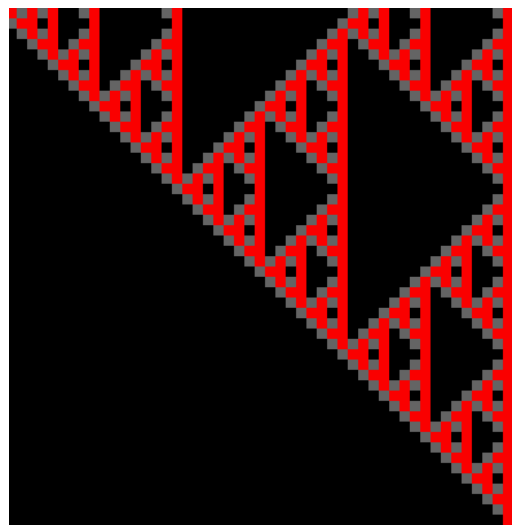


F_1

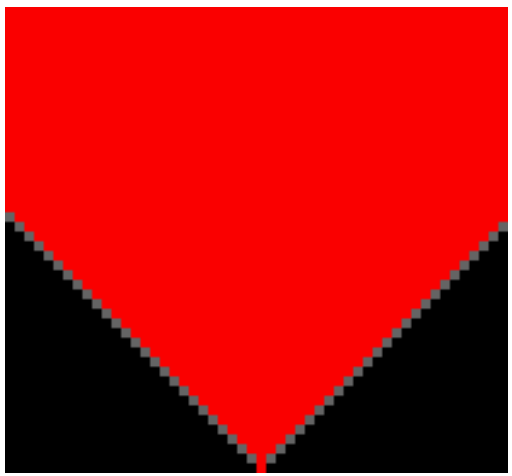


F_2

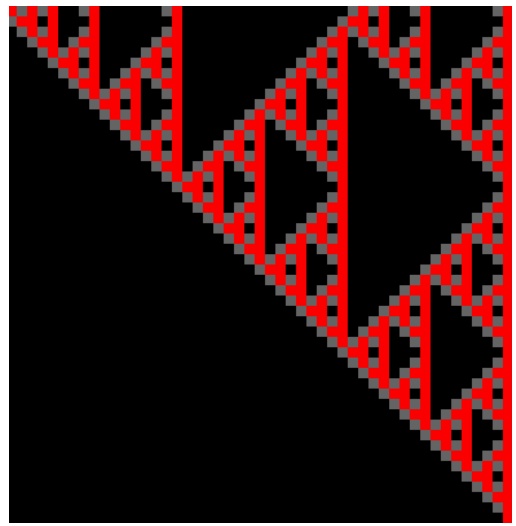
Proof Tool: Dependencies

- δ_z^t: how cell z at time t depends on cell 0 at time 0

$$\delta_z^t : q \mapsto F^t({}^\omega 0 q 0^\omega)_z$$



F_1



F_2

- k-isolated dependencies: δ_z^t **bijective** / $\delta_{z-1}^t, \ldots, \delta_{z-k}^t$ **null**

$$\Rightarrow F^t({}^\omega 0 u 0^\omega)_z \neq 0 \text{ when } |u| \leq k \text{ and } u_0 \neq 0$$

A strongly randomizing/diffusive CA

$$F_2(c)_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot c_z + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot c_{z+1}$$

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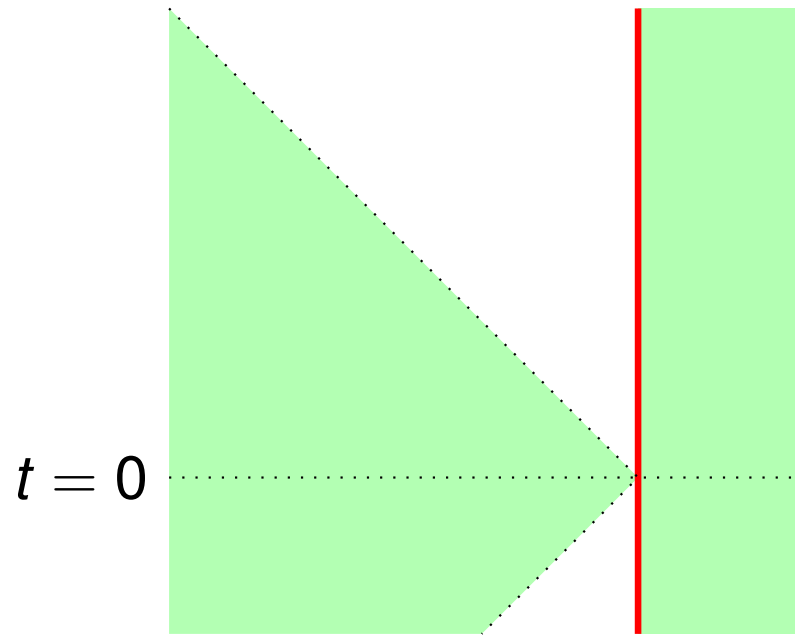
Known dependencies:



red $=$ bijective / green $=$ null

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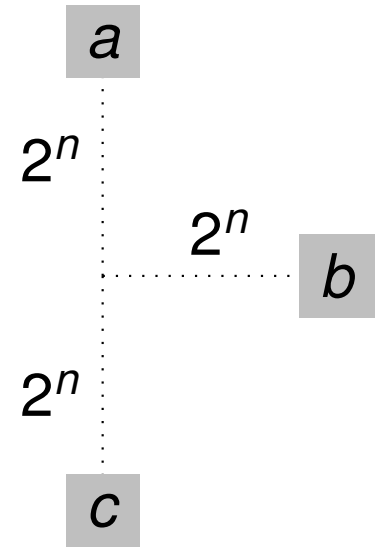
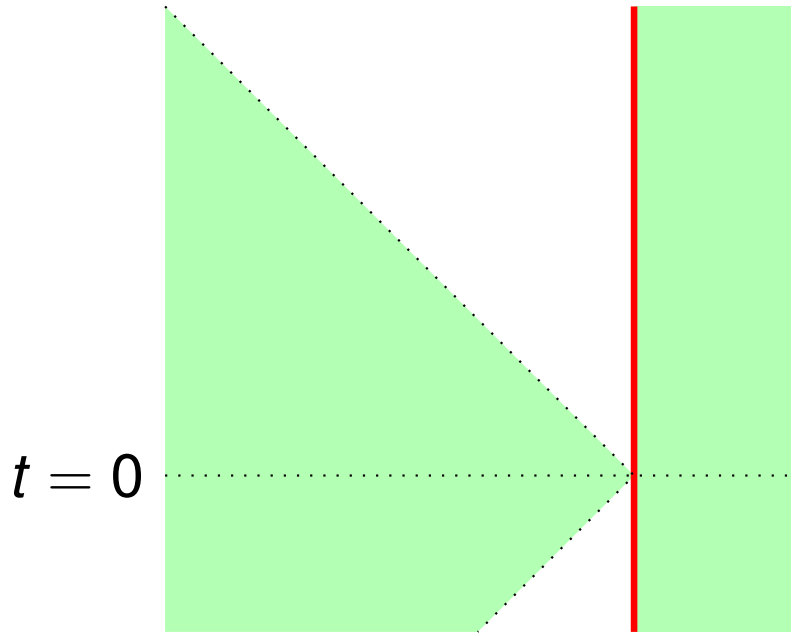


$a \oplus b \oplus c = 0$



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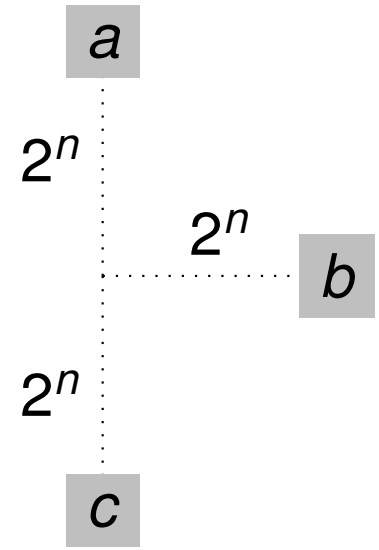

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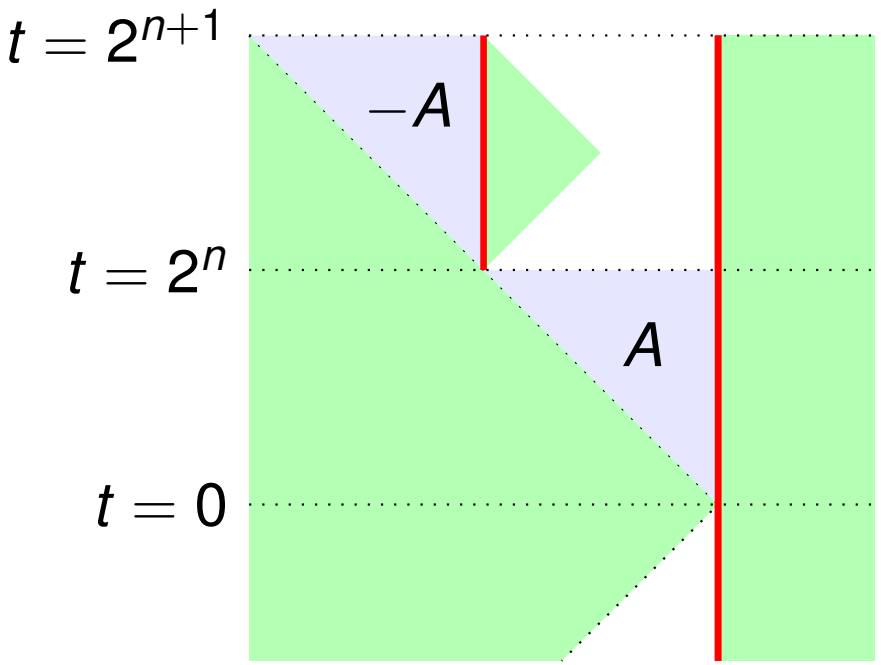

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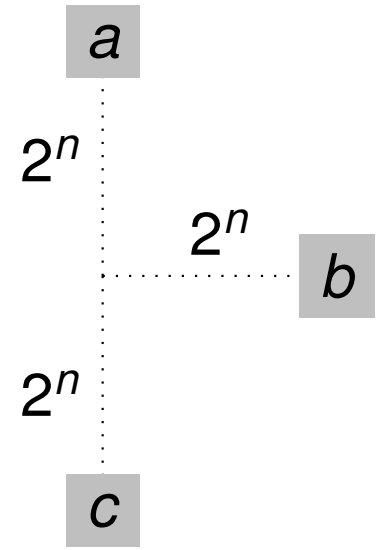

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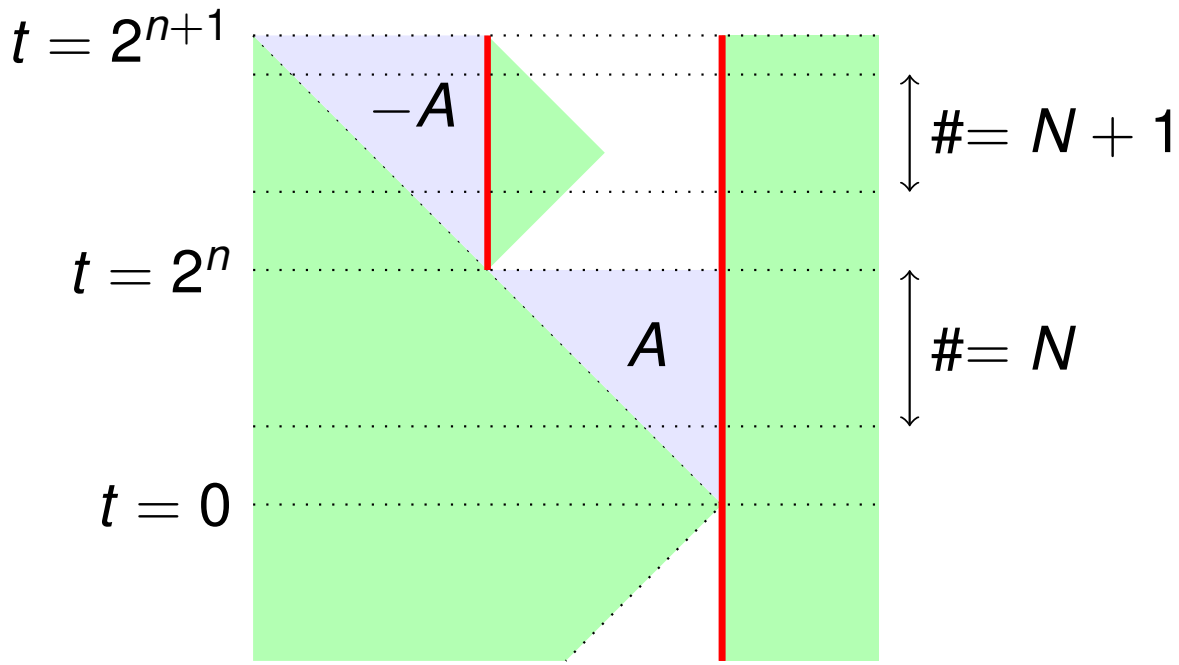

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- *k*-separated dependencies

A strongly randomizing/diffusive CA

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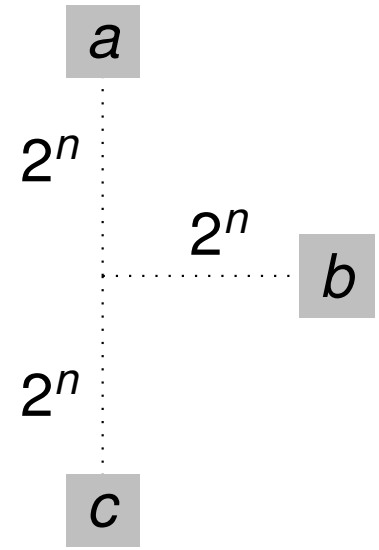
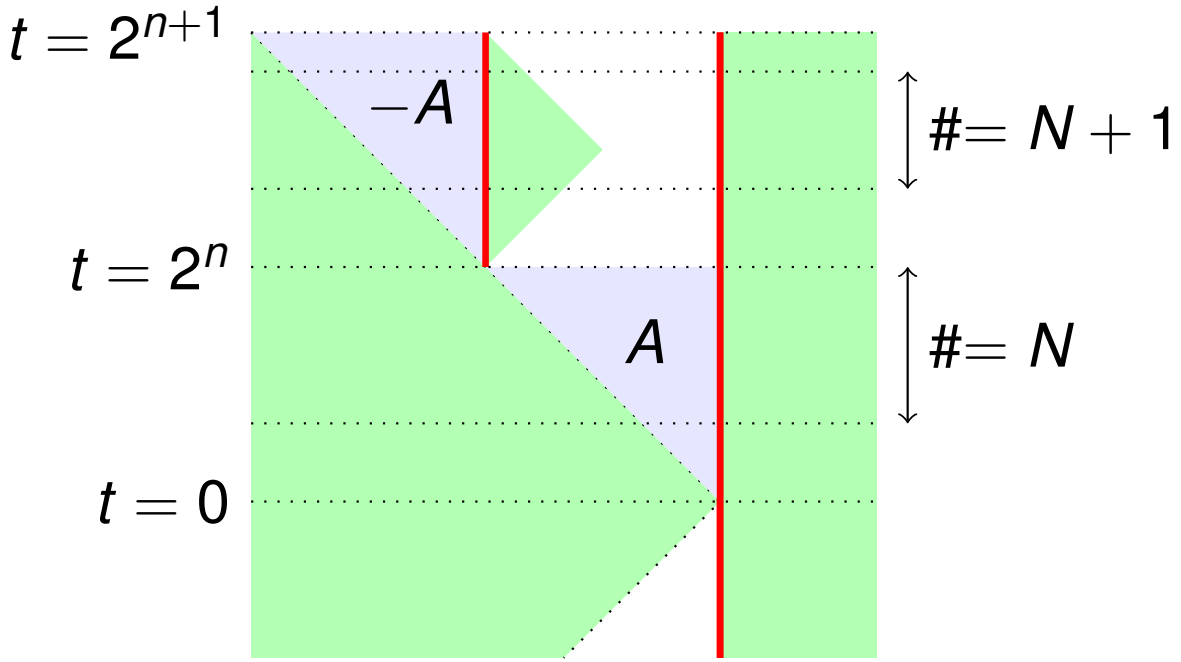
Known dependencies:



$$a \oplus b \oplus c = 0$$



red $=$ bijective / green $=$ null

- ***k*-separated dependencies** + reversibility!

“Super-Abelian” case

- Abelian $F(c)_z = \bigoplus_{i \in V} h_i(c_{z+i})$ with h_i commuting

- All previously cited results are in this case.

“Super-Abelian” case

- Abelian $F(c)_z = \bigoplus_{i \in V} h_i(c_{z+i})$ with h_i commuting
- All previously cited results are in this case.
- $Q = \mathbb{Z}_{p^l}^k$
- **lemma:** $F^{p^{n+l-1}}(c)_z = \left(\sum_{i \in V} h_i^{p^n}(c_{z+ip^n}) \right)^{p^{l-1}}$
- **corollary 1:** no strong randomization
- **corollary 2:** decision algorithm for randomization

Thank you!