Propagation, Diffusion and Randomization in Cellular Automata

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AUTOMATA 2016, Zürich

This talk

- based on 2 joint works
  - **pre-expansivity:** A. Gajardo, V. Nesme
  - **randomization:** B. Hellouin de Menibus, V. Salo
- *general setting*: deterministic 1D cellular automata

Pairs of orbits / orbits of pairs

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- consider some CA

  e.g. **ECA 54**

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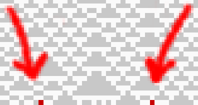

Pairs of orbits / orbits of pairs

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  e.g. **ECA 54**

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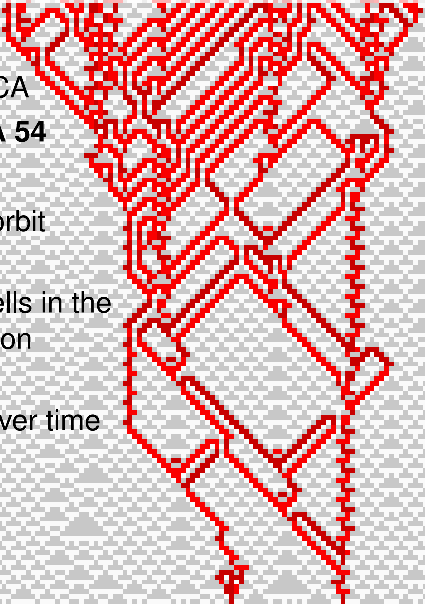
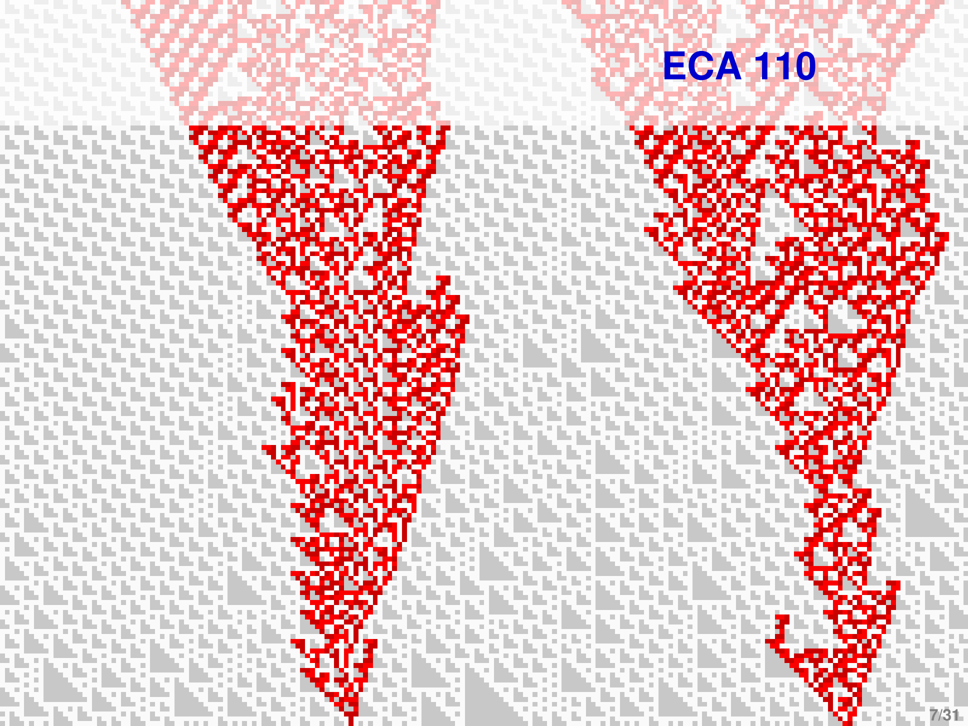
- change some cells in the initial configuration

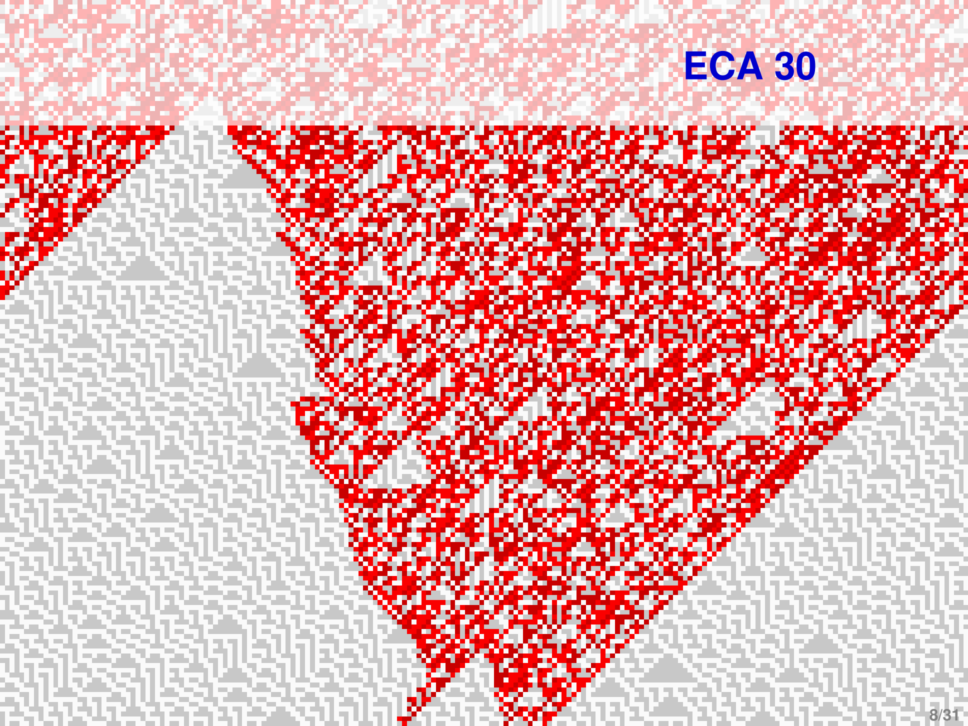
- trace changes over time



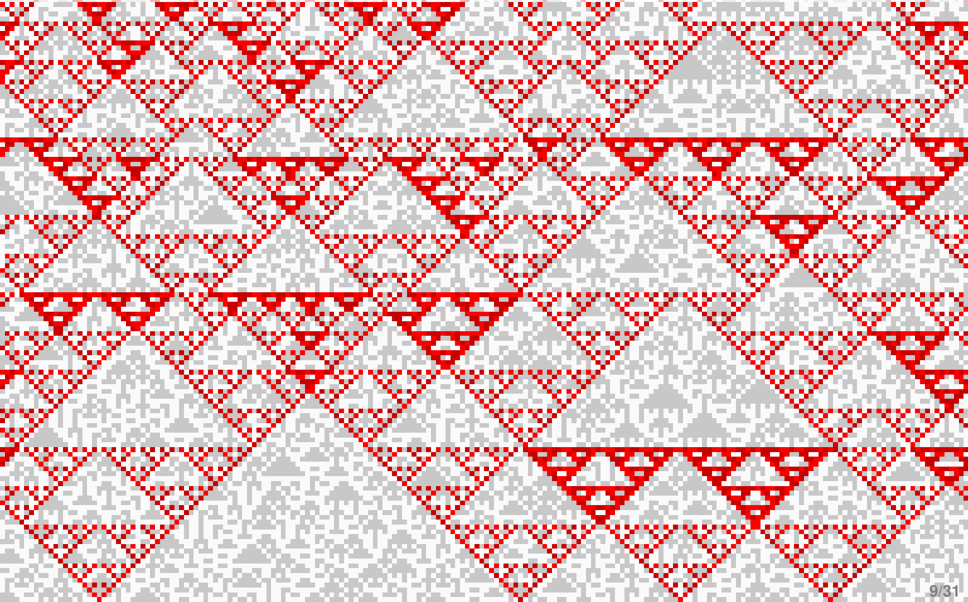
ECA 110



ECA 30



ECA 90



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Proposition **(folklore?, see Kari's course notes)**

No CA can be both **reversible** and **positively expansive.**

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- back to ECA 30

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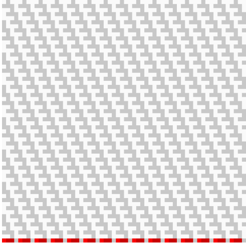

not reversible

Pairs of orbits / orbits of pairs

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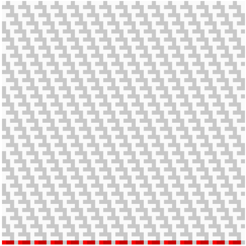

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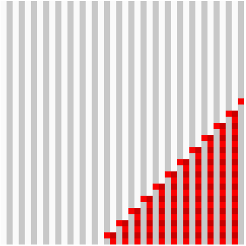

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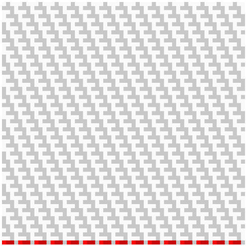

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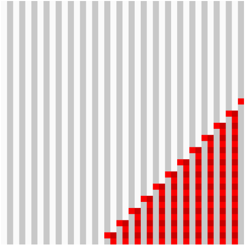

not pos. expansive

Definition: diamond

$c \diamond d \overset{def}{\Longleftrightarrow} c \neq d$ and $\Delta(c, d)$ finite.

Diamonds are forever

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- **surjectivity** $\stackrel{def}{\Longleftrightarrow} \forall c, \exists d : F(d) = c$
- **pre-injectivity** $\stackrel{def}{\Longleftrightarrow} \forall c, d : c \diamond d \Rightarrow F(c) \diamond F(d)$

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A CA is surjective **if and only if** it is pre-injective.

- **Corollary:** reversible CA are surjective (surjunctivity)
- (2010): Garden of Eden Theorem over $Q^G \Leftrightarrow G$ amenable
- Gottschalk problem (1973): is there a group which is not surjunctive?

Dynamic of Diamonds

Dynamic of Diamonds

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  $c \diamond d$ such that $\Delta(F^t(c), F^t(d)) \subseteq [\alpha t - \beta, \alpha t + \beta]$ for all t

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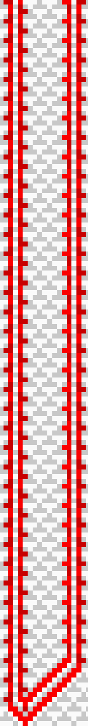
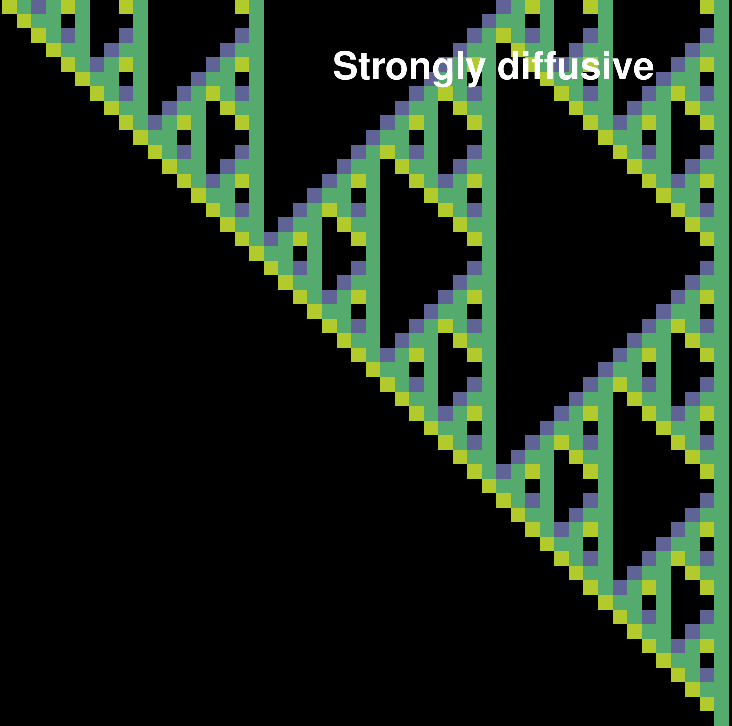
- **pre-expansivity:** *expansivity over diamonds*

$$\exists W \underset{\text{finite}}{\subseteq} \mathbb{Z}, \forall c, d, \; : \; c \diamond d \Rightarrow \exists t, W \cap \Delta(F^t(c), F^t(d)) \neq \emptyset$$

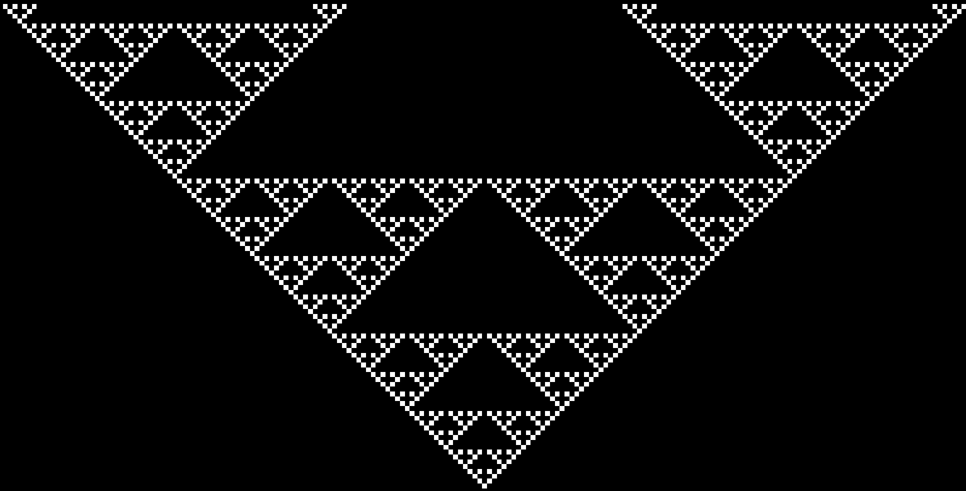
Glider



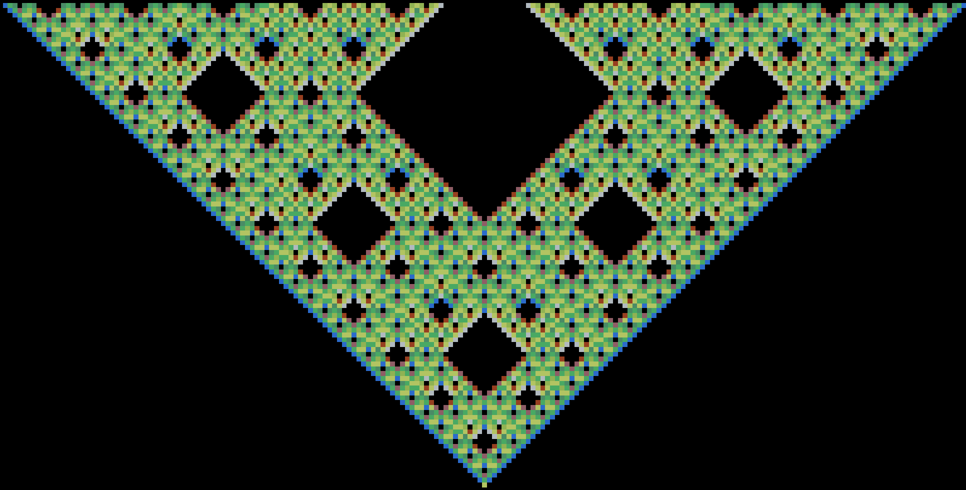


Strongly diffusive

Diffusive but not strongly



Pre-expansive and reversible



Inside Surjectivity

strong diffusivity	$\Rightarrow$	diffusivity		
		$\Downarrow$		
		no glider	$\Rightarrow$	**surjectivity**
		$\Uparrow$		
pos. expansivity	$\Rightarrow$	pre-expansivity		$\Uparrow$
$\Downarrow$				
irreversibility				reversibility

Statistical Equilibrium

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- cylinder sets

$$[u]_z = \{c \in Q^{\mathbb{Z}} : c_z \cdots c_{z+|u|-1} = u\}$$

- translation-invariant **probability measure** μ :

$$\mu([u]_z) = \mu([u]_{z'}) = \mu([u]) \in [0, 1]$$

- example: uniform Bernoulli measure, $\mu_0([u]) = \#Q^{-|u|}$

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Theorem (Hedlund, 1969)

F surjective **if and only if** $F\mu_0 = \mu_0$

Randomization

- **NDB**: **N**on-**D**egenerate **B**ernoulli measure μ

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- *not addressed here: replace NDB by "harmonically mixing"*

Some Results on Randomization

- Miyamoto 1979, Lind 1984
  XOR CA (ECA 102) is randomizing (NDB measures).

- Ferrari-Maass-Martinez-Ney 2000
  $F(c)_z = \alpha c_z + \beta c_{z+1} \bmod p^l$, $\alpha, \beta \neq 0 \bmod p$
  is randomizing (measures with quick correlation decay).

- Pivato-Yassawi 2002
  $F(c)_z = \sum_{j \in J} \alpha_j c_{z+j}$ on Abelian group + conditions on α_j
  is randomizing (harmonically mixing measures).

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- **key point:** enough to consider diamonds with $c = {}^{\omega}0^{\omega}$
  - diamond $\leftrightarrow$ finite configuration
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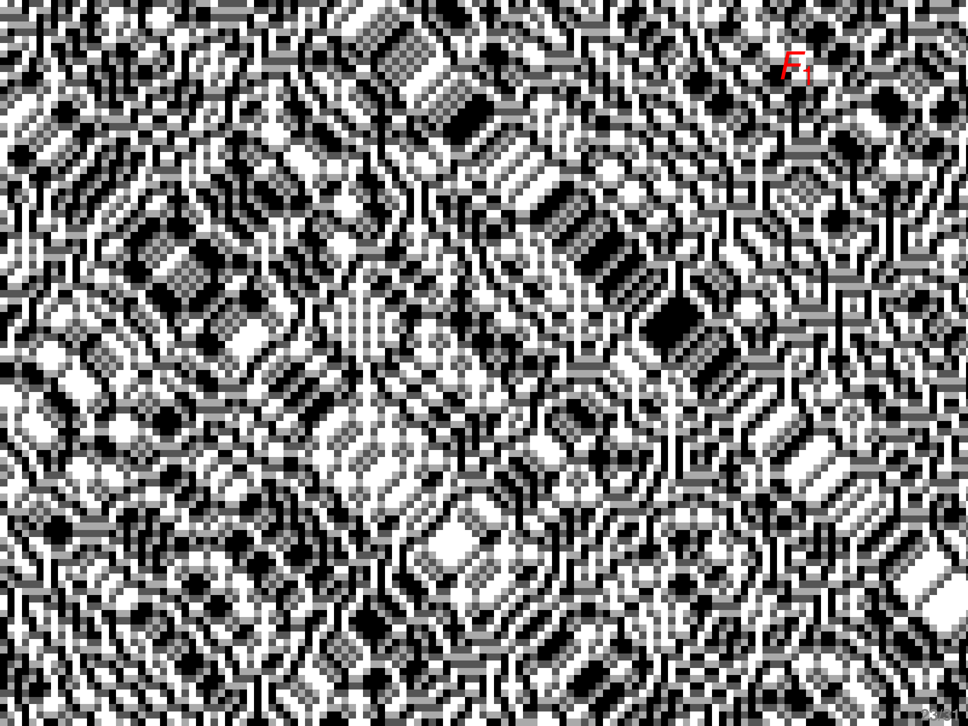
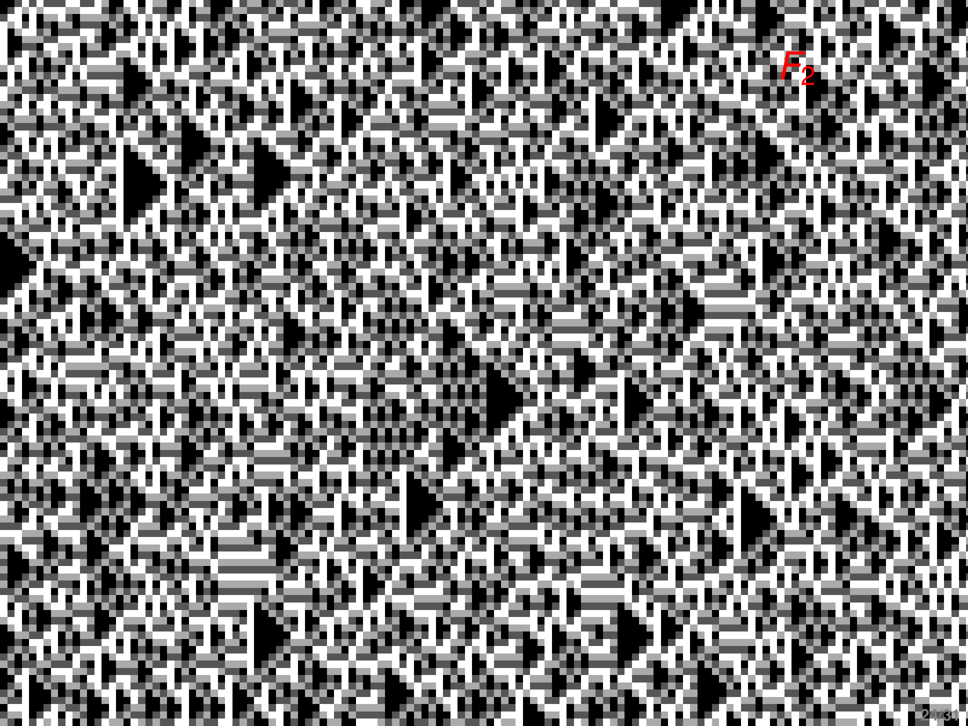
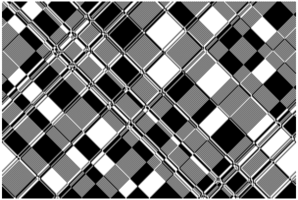
- examples with $(Q, \oplus) = \mathbb{Z}_2^2$
  1. $F_1(c)_z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot c_{z-1} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot c_z + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot c_{z+1}$
  2. $F_2(c)_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot c_z + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot c_{z+1}$

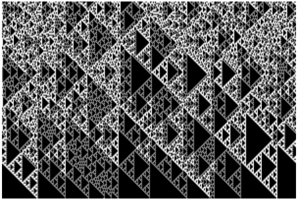

F_1


F2

A Dichotomy Theorem



Having a glider...



...or not.

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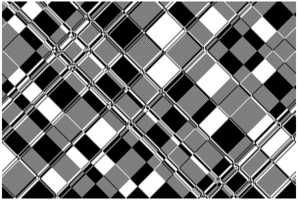

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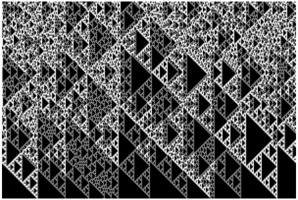

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Theorem

Let F be a $\mathbb{Z}^l_{p^k}$-Abelian CA, then the following are equivalent:

1. F has no glider
2. F is diffusive
3. F is randomizing

For $\mathbb{Z}_{p^l}^k$-Abelian CA:

1. F and G randomizing $\Rightarrow F \times G$ randomizing
2. F randomizing and reversible $\Rightarrow F^{-1}$ randomizing
3. F pre-expansive $\Rightarrow F$ randomizing
4. F randomizing $\Rightarrow$ all its subautomata are randomizing
5. randomizing μ NDB $\Leftrightarrow$ randomizing μ harmonically mixing

A Dichotomy Theorem

Corollaries

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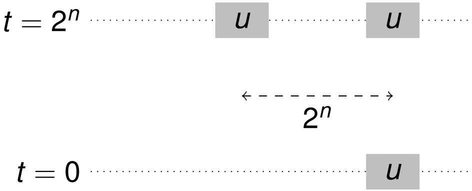
Are the above statements still true for non-Abelian CA?

- XOR has a glider?

A Dichotomy Theorem

Application

- XOR has a glider? **NO!** $F^{2^n}(c)_z = c_z \oplus c_{z+2^n}$



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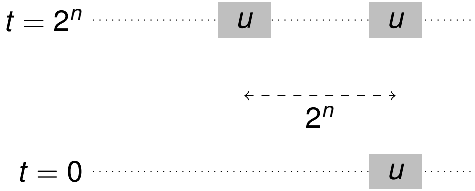

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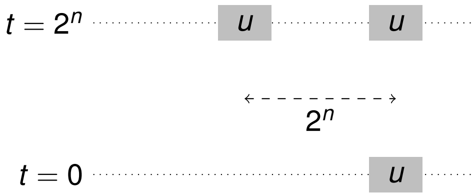

- $\mathbb{Z}_{p^l}^k$ -Abelian $F(c)_z = \bigoplus_{i \in V} h_i(c_{z+i})$ with h_i commuting
- **lemma:** $F^{p^{n+l-1}}(c)_z = \left(\sum_{i \in V} h_i^{p^n}(c_{z+ip^n})\right)^{p^{l-1}}$
- **corollary 1:** decision algorithm for randomization
- **corollary 2:** no strong randomization

A Dichotomy Theorem

About the Proof

1. no glider $\Leftrightarrow$ diffusivity

2. harmonic analysis (inspired from Pivato-Yassawi)
   - characters: $\chi : Q^{\mathbb{Z}} \to \mathbb{C}$ continuous group morphism
   - Fourier coefficients: $\mu[\chi] = \int \chi d\mu$
   - $F\mu[\chi] = \mu[\chi \circ F]$
   - dual F^*: action of F on characters

3. F^* diffusive $\Leftrightarrow$ F randomizing
   (harmonically mixing measures)

4. F^* has a glider $\Leftrightarrow$ F has a glider

Proof Tool: Dependencies

- δ_z^t: how cell z at time t depends on cell 0 at time 0

$$\delta_z^t : q \mapsto F^t({}^\omega 0 q 0^\omega)_z$$

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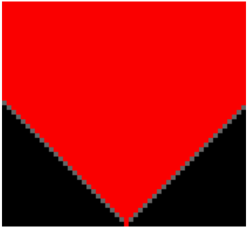

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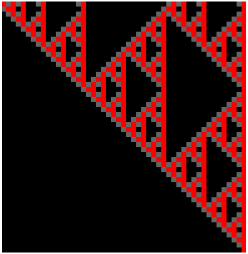

F_2

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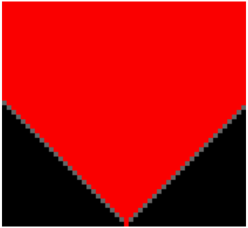

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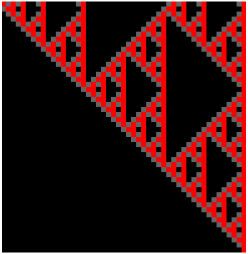

F_2

- k-isolated dependencies: δ_z^t **bijective** / $\delta_{z-1}^t, \dots, \delta_{z-k}^t$ **null**
  $\Rightarrow F^t({}^\omega 0 u 0^\omega)_z \neq 0$ when $|u| \leq k$ and $u_0 \neq 0$

A strongly randomizing/diffusive CA

$$F_2(c)_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot c_z + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot c_{z+1}$$

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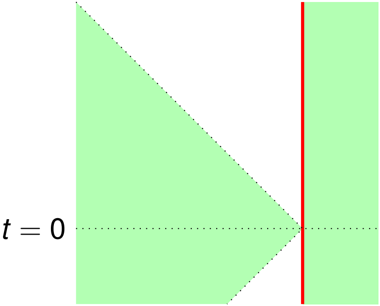
Known dependencies:



red = bijective / green = null

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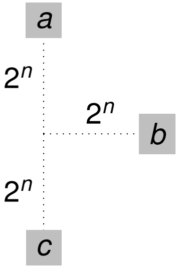
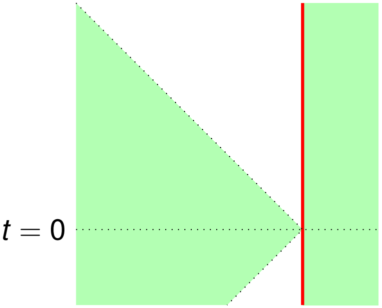


$a \oplus b \oplus c = 0$

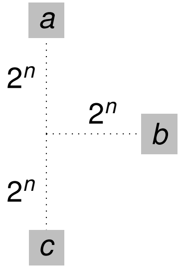


red = bijective / green = null

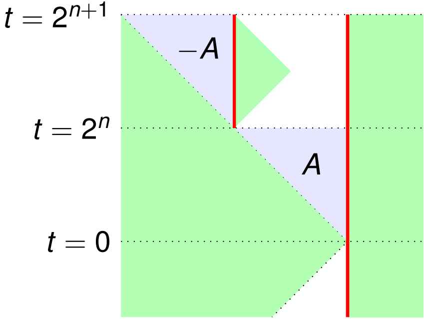
A strongly randomizing/diffusive CA

$$F_2(c)_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot c_z + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot c_{z+1}$$

Known dependencies:



$a \oplus b \oplus c = 0$

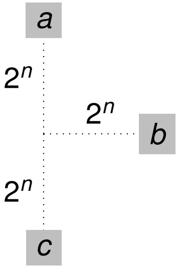


red = bijective / green = null

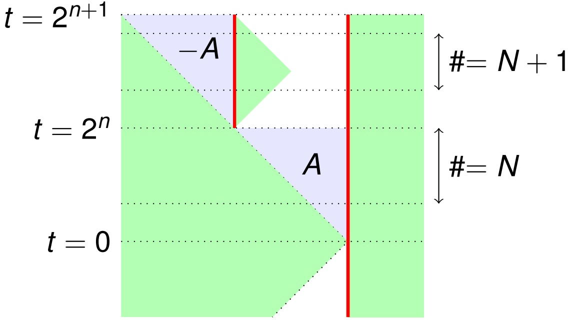
A strongly randomizing/diffusive CA

$$F_2(c)_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot c_z + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot c_{z+1}$$

Known dependencies:



$a \oplus b \oplus c = 0$



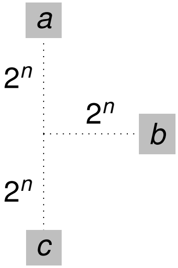
red = bijective / green = null

- ***k*-separated dependencies**

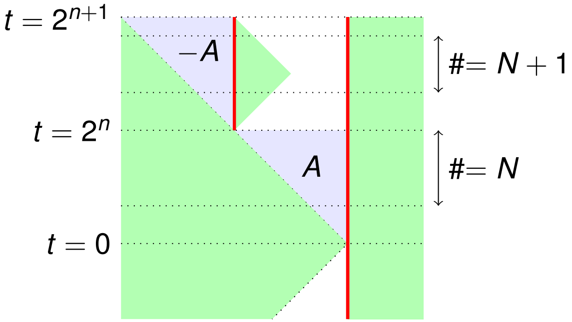
A strongly randomizing/diffusive CA

$$F_2(c)_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot c_z + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot c_{z+1}$$

Known dependencies:



$a \oplus b \oplus c = 0$



red = bijective / green = null

- ***k*-separated dependencies** + reversibility!

Thank you!