Pre-Expansivity in Cellular Automata

Workshop on Symbolic Dynamics on Groups

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1 — Definitions

- G a finitely generated group
- A a finite set
- $F : A^G \to A^G$ a cellular automaton
- $X \subseteq A^G$ a subshift
- $F(X) \subseteq X$

Positive Expansivity

Definition

(X, F) expansive if there is a finite window $W \subseteq G$ such that

$$x \neq y \in X \;\Rightarrow\; F^t(x)_{|W} \neq F^t(y)_{|W} \text{ for some } t$$

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Theorem

(X, F) can not be expansive if we have one of the following:

1. F is reversible and X infinite
2. $h(X) > 0$ and G has superlinear growth

M. A. Shereshevsky (1993), M. Pivato (2013)

Positive expansiveness versus network dimension in symbolic dynamical systems

Pre-Expansivity

- asymptotic equality: $x \overset{\infty}{=} y$ if

$$\{g \in G : x(g) \neq y(g)\} \text{ is finite}$$

Definition

(X, F) pre-expansive if there is a finite $W \subseteq G$ such that

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- expansivity $\Rightarrow$ pre-expansivity
- pre-expansivity $\Rightarrow$ pre-injectivity ($\Rightarrow$ surjectivity)$^{\text{dep. on G}}$
- pre-expansivity $\Rightarrow$ sensitivity to initial conditions

Linear CA

- endow states with a group structure: $(A, \oplus)$
- neutral element denoted 0
- extend $\oplus$ to A^G pointwise
- F $\oplus$-linear if

$$F(x \oplus y) = F(x) \oplus F(y)$$

- a configuration x is **finite** if $x \overset{\infty}{=} \overline{0}$
- then F is **pre-expansive** iff there is a finite $W \subseteq G$ s.t.

$$x \neq \overline{0} \text{ finite } \Rightarrow F^t(x)_{|W} \neq \overline{0} \text{ for some } t$$

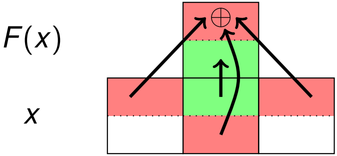
2 — Results

- $G = \mathbb{Z}^d$
- $X = A^{\mathbb{Z}^d}$

An example on $\mathbb{Z}$

- $A = \mathbb{Z}_3 \times \mathbb{Z}_3$
- $\oplus$ componentwise addition modulo 3
- $\pi_1, \pi_2 : A \to \mathbb{Z}_3$ projections on each component
- F defined by

$$F(x)_z = \big(\pi_2(x_z) \oplus \pi_1(x_{z+1}) \oplus \pi_1(x_{z-1}), \pi_1(x_z)\big)$$



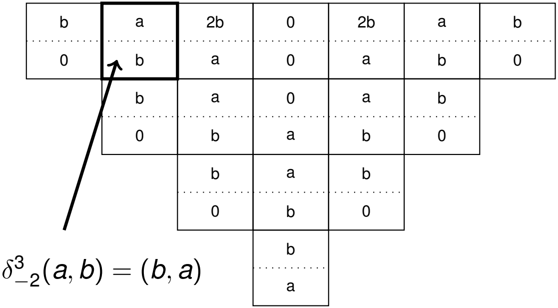
- F is $\oplus$-linear
- F is reversible and hence not positively expansive

An example on $\mathbb{Z}$

- *how cell z at time t depend on cell 0 at time 0?*
- dependency functions $\delta_z^t : A \to A$

$$\delta_z^t(\alpha) = F^t(x_\alpha)_z$$

where $x_\alpha(0) = \alpha$ and $x_\alpha(z \neq 0) = 0$



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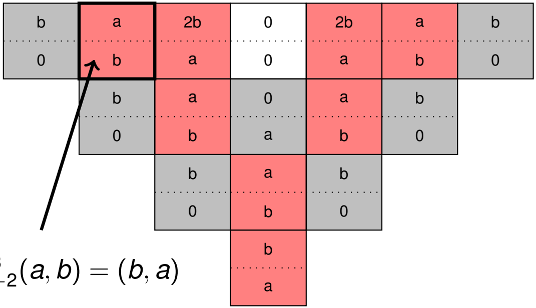
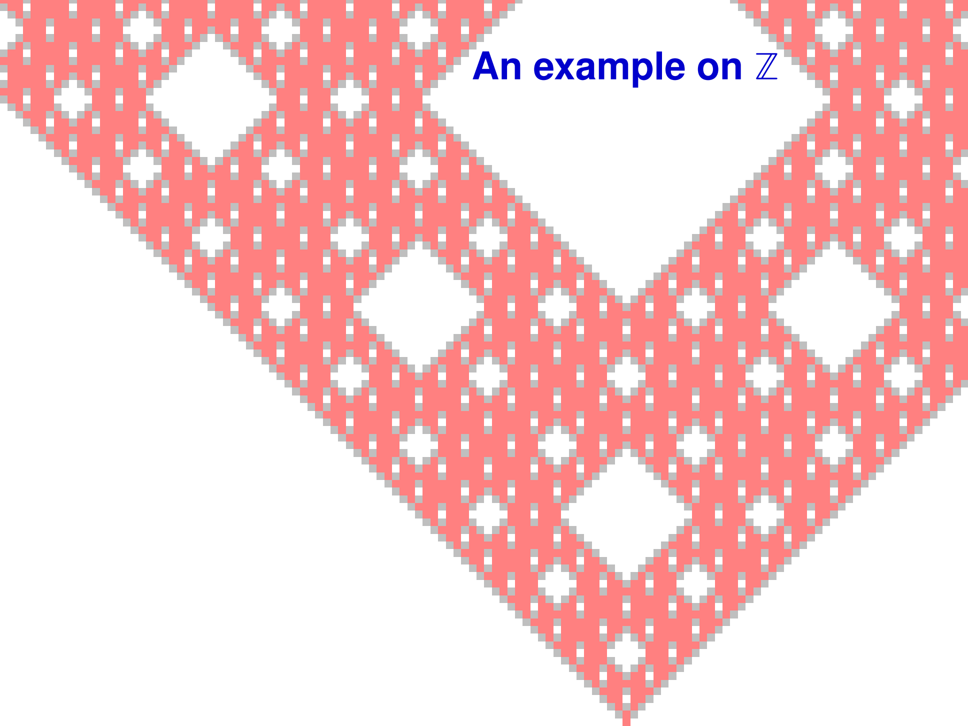


$\delta^3_{-2}(a, b) = (b, a)$

- by linearity: $F^t(x)_z = \bigoplus_{i \in \mathbb{Z}} \delta_i^t\big(x(z - i)\big)$

An example on $\mathbb{Z}$



An example on $\mathbb{Z}$

- k-semi-isolated dependencies:

$$X_k^+ = \left\{ z : \exists t, \begin{cases} \delta_z^t \text{ bijection and} \\ \delta_{z+i}^t = 0, \forall i \in [1, k] \end{cases} \right\}$$

$$X_k^- = \left\{ z : \exists t, \begin{cases} \delta_z^t \text{ bijection and} \\ \delta_{z-i}^t = 0, \forall i \in [1, k] \end{cases} \right\}$$

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Lemma

If $X_k^+ \cup X_k^-$ is neither lower-bounded nor upper-bounded $\forall k$, then F is pre-expansive.

- proving the hypothesis of the Lemma on our example:

$$F^{2\cdot 3^m} = \sigma_{-3^m} \circ F^{3^m} \oplus \sigma_{3^m} \circ F^{3^m} \oplus Id$$

Looking for Examples on $\mathbb{Z}^2$

- $A = \mathbb{Z}_2$
- $V =$ von Neumann neighborhood
- $F(X)_0 = \sum_{i \in V} x_i \bmod 2$

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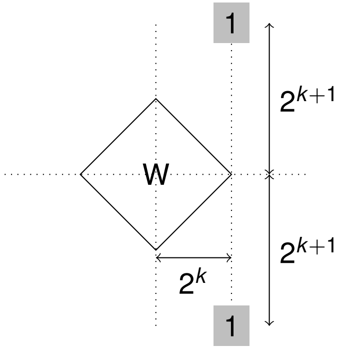

Stop Looking for (linear) Examples on $\mathbb{Z}^2$

Theorem

There is no pre-expansive CA on $\mathbb{Z}^d$, $d \geq 2$, which is linear for an Abelian group structure on states.

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Lemma (traces determined by prefix of linear size)

For such a linear CA there is $\lambda(n) \in O(n)$ s.t. if

1. $c(z) = c'(z) = 0, \forall z$ s.t. $\|z\|_\infty > n$
2. $F^t(c)_{|W} = F^t(c')_{|W}, \forall t \leq \lambda(n)$

then $\forall t, F^t(c)_{|W} = F^t(c')_{|W}$.

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then $\forall t, F^t(c)_{|W} = F^t(c')_{|W}$.

- uses self-similarity structure of dependency functions δ_z^t

J. Gütschow, V. Nesme, R. F. Werner (2010)

The fractal structure of Cellular Automata on Abelian Groups

Looking closer: k-expansivity

- exactly k differences: $x \neq_k y$ if

$$\#\{g \in G : x(g) \neq y(g)\} = k$$

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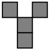
- pre-expansive $\Rightarrow$ k-expansive for all k
- converse true for $\mathbb{Z}$ and $\mathbb{Z}_p$-linear CA on $\mathbb{Z}^d$
- k-expansive $\Rightarrow$ sensitivity to initial conditions
- does not imply pre-injectivity

Back to Examples on $\mathbb{Z}^2$

- $A = \mathbb{Z}_2$
- $V_1 =$ Cole neighborhood 
- $F(X)_0 = \sum_{i \in V_1} x_i \bmod 2$

Fact

F is not 1-expansive.

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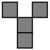
- $V_1 =$ Cole neighborhood 
- $F(X)_0 = \sum_{i \in V_1} x_i \bmod 2$

Fact

F is not 1-expansive.

- $V_2 =$ von Neumann neighborhood
- $G(X)_0 = \sum_{i \in V_2} x_i \bmod 2$

Fact

G is 3-expansive.

- substitution describing W-trace + case analysis

Questions?