Introducing freezing cellular automata

Taller @ Concepción

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Forewords

- everything in 2D
- work in progress
- joint research with E. Goles

Threshold rules

- alphabet $Q = \{0, 1\}$
- neighborhood N arbitrary
- rule with **threshold** θ

$$F(x)_{(0,0)} = \begin{cases} 0 & \text{if } x_{(0,0)} = 0 \text{ and } \#\{z \in N : x_z = 1\} < \theta \\ 1 & \text{else} \end{cases}$$

- limit fixed point

$$F^{\infty}(x) = \lim_{t \to \infty} F^t(x)$$

- fix $0 \leq p \leq 1$
- Bernoulli distribution: 1 with probability p
- Bootstrap percolation **happens**, denoted $\mathbb{B}(p)$, if

$$\mu\left(\{x : F^{\infty}(x) = 1^{\mathbb{Z}^2}\}\right) = 1$$

- **Lemma:** if $p \leq p'$ then $\mathbb{B}(p) \Rightarrow \mathbb{B}(p')$

A central question

What is the critical probability $\inf\{p : \mathbb{B}(p)\}$?

Threshold rules

Criticality

- 3 behaviors for a threshold rule F:

hypercritical finite seed of 1s

$$\exists x \text{ finite s.t. } F^{\infty}(x) \text{ is infinite}$$

subcritical finite alliance of 0s

$$\exists x \text{ co-finite s.t. } F(x) = x$$

critical other cases

Proposition

- hypercritical $\Rightarrow$ critical probability is 0
- subcritical $\Rightarrow$ critical probability is 1

- criticality is completely determined
  - Gravner-Griffeath 1996
  - simple geometrical/combinatorial conditions on N and θ
  - polynomial algorithm

- a large class of critical rules all behave the same way

Theorem (Duminil-Holroyd,2012)

If F is critical with a “nice” N then critical probability is 0.

Threshold rules

Bootstrap percolation results

- criticality is completely determined
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Theorem (Duminil-Holroyd,2012)

If F is critical with a “nice” N then critical probability is 0.

I don’t know...

Is there a critical F with critical probability $\neq 0$?

Self-assembly tiling

$$(T, s, \tau)$$

- T is a **Wang tileset with strengths**: on each edge
  1. a label (matching condition)
  2. an integer (matching strength)
- $\tau > 0$ is the **temperature**
- an assembly of tiles is **τ-stable** if separating it into 2 parts implies removing a set of matching edges with total strength at least τ
- s is a τ-stable finite assembly
- **dynamics:**
  - starting from s
  - $a \to a'$ by adding a tile to a that preserves τ-stability

- general model Turing-universal in 2D (Winfree, 1998)
- temperature 1

Theorem (Cook-Fu-Schweller, 2011)

Temperature 1 self-assembly is Turing-universal in 3D

A central question

What is the computational power of (2D) temperature 1?

- Warning! **asynchronous systems**
- what is a valid simulation? what is universality?

- Biblio
  - Doty-Lutz-Patitz-Schweller-Summers-Woods-(Meunier-Theyssier)
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- Definition
  1. block encoding
  $$\mathbb{Z}^2 \;\leftrightarrow\; (m\mathbb{Z}) \times (p\mathbb{Z})$$
  2. fuzz condition
  empty cell $\leftrightarrow$ empty block (exception!)

Self-assembly tiling

Simulations and universality

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- Definition
  1. block encoding
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  2. fuzz condition
     empty cell $\leftrightarrow$ empty block (exception!)
  3. requirements:
     - same productions (through encoding)
     - same dynamics (through encoding) not clear!

- with a strong notion of “same dynamics”

Theorem (USA crew, 2012)

There is an intrinsically universal tile set at temperature 2.

- with a strong notion of "same dynamics"

Theorem (USA crew, 2012)

There is an intrinsically universal tile set at temperature 2.

- using only requirement of "same productions"

Theorem (USA crew + Meunier-Theyssier, 2013)

No temperature 1 tile set can be intrinsically universal.


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Freezing cellular automata

- $Q = \{0, \ldots, n-1\}$ with natural order $\leq$
- N arbitrary neighborhood
- F is **freezing** if
$$\forall x, \forall z : \; F(x)_z \geq x_z$$

Freezing cellular automata

- $Q = \{0, \ldots, n-1\}$ with natural order $\leq$
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- F is **freezing** if

$$\forall x, \forall z : \; F(x)_z \geq x_z$$

- limit fixed point always defined

$$F^{\infty}(x) = \lim_{t \to \infty} F^t(x)$$

- *special case:* F is **monotonic** wrt $\leq$
- *special case:* F is **outer-multiset** if invariant by permutation of outer neighbors

Freezing cellular automata

- threshold CA are **monotonic outer-multiset freezing**
- self-assembly systems are **freezing** (asynchronous) CA
- other examples (**outer-multiset freezing**):
  - infection propagation (SIR models)
  - life without death

Freezing cellular automata

Predictability

- kind of poor dynamically (convergence to fixed-point)

Freezing cellular automata

Predictability

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- ...but P-complete for prediction problem,
  - embed any 1D P-complete CA
  - life without death is P-complete (???)

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- ...but prediction has low communication complexity

Proposition

The prediction problem of any freezing CA has communication complexity in $O(n \log(n))$

recall: for a general CA this can be $\Omega(n^2)$

Freezing cellular automata

Bootstrap percolation

- **problem:** probability that $F^{\infty}(x)$ is uniformly $n - 1$?

- **problem:** probability that $F^{\infty}(x)$ is uniformly $n-1$?
- generalized definition of criticality:

hypercritical
: finite seed generates infinitely many $n-1$ in any context

subcritical
: finite alliance of states $\neq$ from $n-1$ resistant to any context

critical
: other cases

Freezing cellular automata

Bootstrap percolation

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finite seed generates infinitely many $n-1$
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Proposition

Each property among hypercriticality, subcriticality and criticality is undecidable for freezing CA

Proposition

The general bootstrap percolation problem:

- input: CA + Bernouilli distribution
- question: percolation with probability 1?

is also **undecidable**.

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Conjecture

- Hypercriticality, subcriticality and criticality are **decidable** for monotonic outer-multiset freezing CA

Freezing cellular automata

Bootstrap percolation

Proposition

The general bootstrap percolation problem:

- input: CA + Bernouilli distribution
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Conjecture

- Hypercriticality, subcriticality and criticality are **decidable** for monotonic outer-multiset freezing CA

- Questions
  1. same problem with monotonic freezing CA?
  2. same problem with outer-multiset freezing CA?
  3. critical rules with non-zero critical probability?
  4. non-monotonicity of the percolation probability?

Freezing cellular automata

Simulation and universality

- how rich are the dynamics of freezing CA?
- are freezing CA richer than self-assembly tilings?
- **key point:** max number of changes per cell

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- are freezing CA richer than self-assembly tilings?
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Proposition

No 1-change von Neumann freezing CA can simulate all freezing CA.

- **Corollary:** self-assembly tilings cannot simulate all freezing behaviors

Conjectures

1. there is a **2-changes von Neumann** freezing CA universal for **all freezing** CA
2. there is a **1-change** freezing CA of larger radius which is universal for **all freezing** CA
3. there is a **1-change von Neumann** freezing CA universal for all **1-change von Neumann** freezing CA and it can be chosen as a self-assembly tiling

- **also:** outer-multiset? monotonicity? 3D? clean definitions of simulations? etc