Freezing Cellular Automata and Bootstrap Percolation

UAI - Doctorado en Ingeniería de Sistemas Complejos

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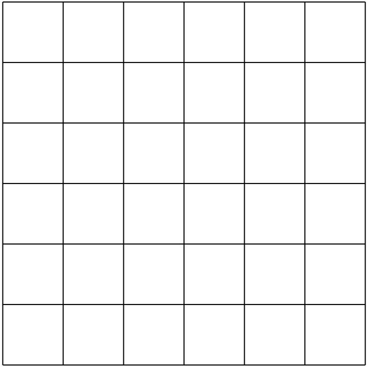
Forewords

- joint research with E. Goles, N. Ollinger, V. Salo, I. Törmä
- partly work in progress

Cellular automata?

Discrete, discrete, discrete...



1. discrete space

Cellular automata?

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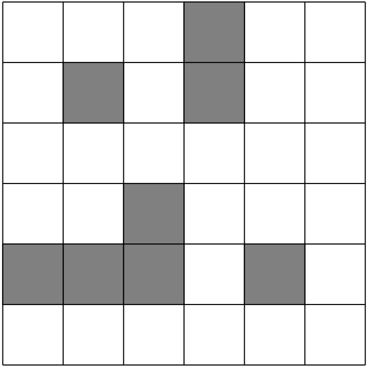

1. discrete space
2. finite set of local states

Cellular automata?

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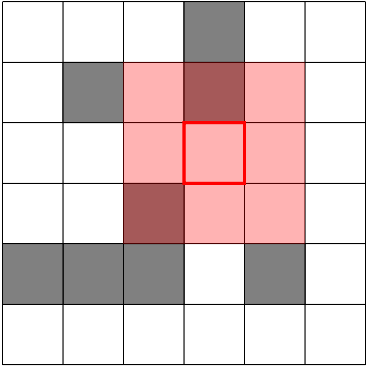
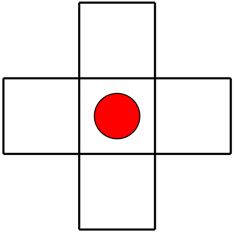


1. discrete space
2. finite set of local states
3. uniform local evolution law at discrete time steps

'The' bootstrap percolation CA



- **states:** $\{0, 1\}$
- if $\#1 \geq 2$ in neighb. then become 1
- otherwise don't change

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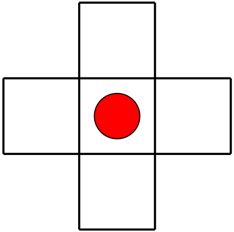

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General questioning

Starting from a random initial configuration...

- what is the probability that **all** cells turn into 1?
- how does this depend on the initial random distribution?

’The’ bootstrap percolation CA

Results

- Chalupa et. al, 1979: introduction and experiments

- Enter, 1987:

  In the **infinite case**, for **any initial density** of 1s, every cell eventually turns into 1 with prob. 1

- Holroyd, 2003:

  In the **finite case** $N \times N$, the **critical initial density** of 1s to obtain percolation with high probability is:

$$\frac{(1 + o(1))\pi^2}{18 \log N}$$

Threshold rules

- states $Q = \{0, 1\}$
- neighborhood N: arbitrary finite subset of $\mathbb{Z}^2$
- rule with **threshold** θ

$$F(x)_z = \begin{cases} 0 & \text{if } x_z = 0 \text{ and } \#\left\{z' \in N : x_{z+z'} = 1\right\} < \theta \\ 1 & \text{else} \end{cases}$$

- limit fixed point

$$F^\infty(x) = \lim_{t \to \infty} F^t(x)$$

Definition (percolation set)

$$P_F = \{x : F^\infty(x) = 1^{\mathbb{Z}^2}\}$$

- μ_p Bernoulli distribution: 1 with probability p, $0 \leq p \leq 1$
- By ergodicity, $\mu_p(P_F) = 0$ or 1
- $\mu_0(P_F) = 0$ and $\mu_1(P_F) = 1$

- **Lemma:** if $p \leq p'$ then $\mu_p(P_F) \leq \mu_{p'}(P_F)$

Central question

What is the critical probability $p_c(F) = \inf\{p : \mu_p(P_F) = 1\}$?

- 3 behaviors for a threshold rule F:

supercritical finite seed of 1s

$\exists x$ finite s.t. $F^{\infty}(x)$ is infinite

subcritical finite alliance of 0s

$\exists x$ co-finite s.t. $F(x) = x$

critical other cases

Proposition

- supercritical $\Rightarrow p_c(F) = 0$
- subcritical $\Rightarrow p_c(F) = 1$

- (sub/super)criticality is completely determined for symmetric N (*i.e.* $N = -N$)
  - Gravner-Griffeath 1996
  - simple geometrical/combinatorial conditions on N and θ
  - polynomial algorithm

- a large class of critical rules all behave the same way

Theorem (Gravner-Griffeath,1996)

If F is critical with N symmetric then $p_c(F) = 0$.

Theorem (Duminil-Holroyd,2012)

If F is critical with a "nice" N then finite-size sharp transition is known.

μ-limit sets and μ-nilpotency

- F is any CA, μ any measure

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- iterations of F on μ: $\mu_t(X) = \mu\big(F^{-t}(X)\big)$
- a word u is μ-**persistent** if

$$\mu_t(u) \not\to 0 (\text{when } t \to \infty)$$

- **μ-limit set** Ω_μ = configurations with only persistent words
- F **μ-nilpotent** if Ω_μ is a single (uniform) configuration

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Remark

Questions about bootstrap percolation are questions about μ-nilpotency

μ-limit sets and μ-nilpotency

- in general, question about Ω_μ are **very** hard!

μ-limit sets and μ-nilpotency

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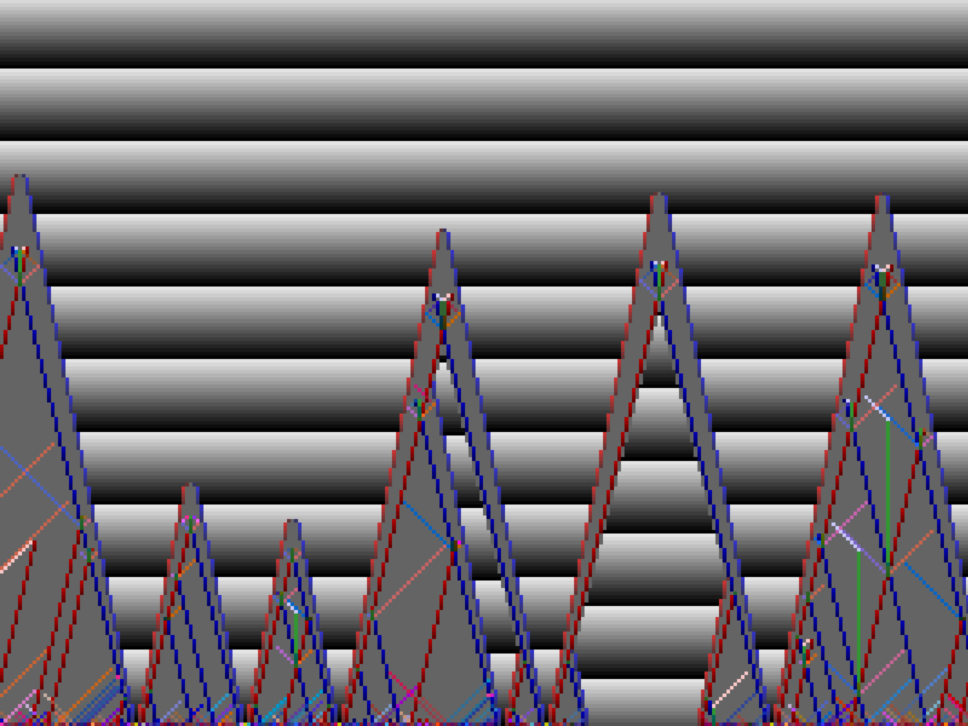
Theorem (Delacourt *et al.*, 2013)

- the μ-nilpotency problem is Π^0_3-complete
- deciding any non-trivial property about Ω_μ is Π^0_3-hard

Proposition (Delacourt *et al.*, 2013)

- There exists F such that $\Omega_\mu(F) \neq \Omega_\mu(F^2)$.
- μ-nilpotency is a simpler problem (Π^0_2) in the simply convergent case



Freezing cellular automata

Freezing cellular automata

- $Q = \{0, \ldots, n-1\}$ with natural order $\leq$
- N arbitrary neighborhood
- F is **freezing** if

$$\forall x, \forall z : \; F(x)_z \geq x_z$$

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- F is **freezing** if

$$\forall x, \forall z : \; F(x)_z \geq x_z$$

- limit fixed point always defined

$$F^{\infty}(x) = \lim_{t \to \infty} F^t(x)$$

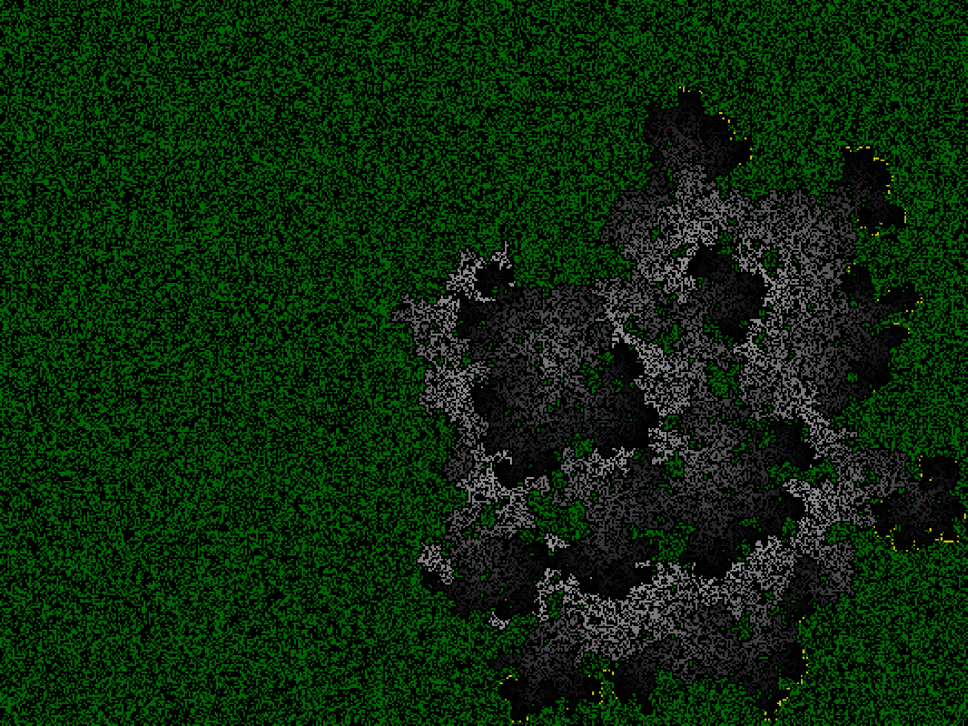
- *special case:* F is **monotonic** wrt cellwise order

$$x \leq^* y \Rightarrow F(x) \leq^* F(y)$$

Freezing cellular automata

Examples

- threshold CA are **monotonic freezing**
- self-assembly systems are **freezing** (asynchronous) CA
- other examples (**freezing**):
  - infection propagation (SIR models)
  - life without death



Freezing cellular automata

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Freezing cellular automata

- kind of poor dynamically (convergence to fixed-point)
- ...but interesting from the point of view of complexity theory

Work in progress with E. Goles and N. Ollinger

- Turing-universality even in 1D
- prediction NL-complete in 1D
- prediction P-complete in 2D
- communication complexity of prediction in $O(n^d \log(n))$ in dimension d
- no intrinsic universality in 1D
- intrinsic universality in 2D
- ...

- The general problem:
  - input: F freezing + Bernouilli distribution μ_p
  - question: is F μ_p-nilpotent?

- **Lemma:** for F freezing the sequence $\left(F^t(\mu_p)\right)_t$ is simply convergent

Freezing cellular automata

Decidability of Bootstrap percolation

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Unpublished result 1 (with V. Salo and I. Törmä)

The general problem above is undecidable, even when restricted to 2-states freezing CA.

Unpublished result 2 (with V. Salo and I. Törmä)

The following problem is **decidable**

- input: F freezing monotone with 2 states
- question: is there p such that F is μ_p-nilpotent?

- decidable reduction to two cases depending on triggering patterns
  1. subcritical rules
  2. a rule that is μ_p-nilpotent for some p (classical directed percolation theory)

Freezing cellular automata

Decidability of Bootstrap percolation

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Conjecture

The general μ-nilpotency problem is decidable for all freezing monotone CA.

Unpublished result 3 (with V. Salo and I. Törmä)

There is a freezing CA F with 2 states, and $0 < p_1 < p_2 < 1$ such that:

- F **is** μ_{p_1}-nilpotent
- F **is not** μ_{p_2}-nilpotent

p	0	p_1	p_2	1
μ_p**-nilpotency**	NO	YES	NO	YES

Freezing cellular automata

Non-monotonic Bootstrap percolation

Unpublished result 3 (with V. Salo and I. Törmä)

There is a freezing CA F with 2 states, and $0 < p_1 < p_2 < 1$ such that:

- F **is** μ_{p_1}-nilpotent
- F **is not** μ_{p_2}-nilpotent

p	0	p_1	p_2	1
μ_p**-nilpotency**	NO	YES	NO	YES

- $F = M \circ D$ where
  - M is monotone
  - D is a 'density detection' rule
  - $F^n = M^n \circ D$
- $D(\mu_p)$ is **not** a Bernoulli dsitribution (correlations), but analysis of M starting from this distribution is possible

Gracias!