Higher Dimensional Dynamics
in Cellular Automata

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(joint work with Mathieu Sablik)

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[1] at the time of writing, CNRS still exists...

Contents of the talk

- **Starting point:**
  - Topological dynamics and 1D CA in the Cantor setting
  - P. Kůrka's classification
- **2 extensions:**
  - Higher dimensions
  - Directional dynamics framework
- **Points of view:**
  - Construction of examples
  - Differences 1D/2D
  - Undecidability

Topological dynamics

Classical properties

- F a continuous function acting on a metric space (X, d)

- **Equicontinuity point** (at x):

$$\forall \epsilon, \exists \delta, \forall y : d(x,y) \leq \delta \Rightarrow \forall t, d\big(F^t(x), F^t(y)\big) \leq \epsilon$$

- **Sensitivity to initial conditions:**

$$\exists \epsilon, \forall x, \forall \delta > 0, \exists y : d(x,y) \leq \delta \text{ and } \exists t, d\big(F^t(x), F^t(y)\big) \geq \epsilon$$

- **(Positive) expansivity:**

$$\exists \epsilon, \forall x, y : x \neq y \Rightarrow \exists t, d\big(F^t(x), F^t(y)\big) \geq \epsilon$$

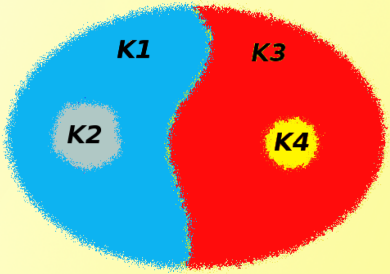
- F a **1D** CA, d the Cantor distance

Proposition (P. Kůrka)

F possesses equicontinuity points $\iff$ F is not sensitive

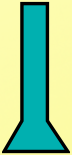
Corollary

*P. Kůrka's dynamical classification for **1D** CA:*

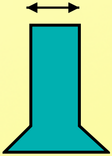


- F a 1D CA with radius r

  - Obstacle



- F a 1D CA with radius r
  - Obstacle
  - Wall: obstacle of width $\geq 2r$

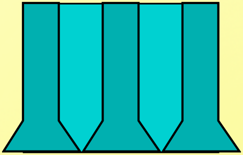


Topological dynamics

Obstacles, walls and wall thickening

- F a 1D CA with radius r

  - Obstacle
  - Wall: obstacle of width $\geq 2r$
  - Concatenation of walls yields a larger wall



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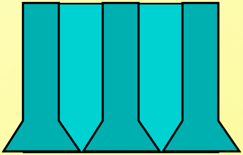

$$F \text{ is sensitive} \iff F \text{ has no wall} \iff F \text{ is } 2^{-2r}\text{-sensitive}$$

Topological dynamics

What about higher dimensions?

- ▷ **key idea:** *in 1D, obstacles are* $(d-1)$*-dimensional objects so they can disconnect the lattice*

Topological dynamics

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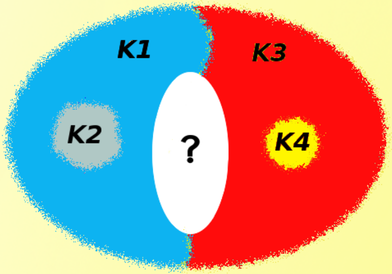
Proposition

There exists a 2D CA without equicontinuity point and which is not sensitive to initial conditions



Non-sensitive and without equicontinuity point

Global description

- 2 parts corresponding to 2 components of the states set:
  - **obstacles:** Q_o
  - **particles:** Q_p

$$Q = Q_o \cup Q_p$$

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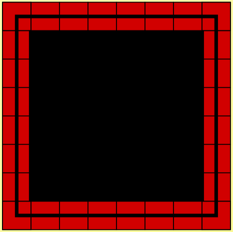
  - finite type condition
  - on error $\rightarrow \square$



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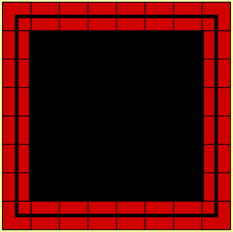
- $Q_p = \{\blacksquare, \blacksquare\}$
  - locally defined dynamics
  - on error $\rightarrow \square$



Proposition

F is not sensitive

Proof.

$\exists$ confs with arbitrarily large obstacles around the center cell. □

Non-sensitive and without equicontinuity point

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Proposition

F has no equicontinuity point

Proof.

- if x is such a point, it must contain some obstacle
- obstacles must be disjoint
- any field of disjoint obstacles can be crossed by particles

□

1D *vs.* 2D

Proposition

The sensitivity constant is not computable when $d \geq 2$.

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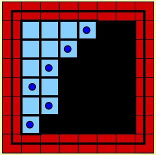
- use the same construction
- put a Turing computation inside obstacles



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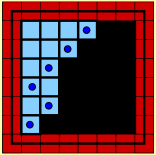

- obstacles are destroyed when a halting state is reached
- Halting TM $\Rightarrow$ bound on obstacles $\Rightarrow$ sensitivity

□

Equicontinuous CA

How large periods can be?

Proposition

F equicontinuous

$\iff$ *F is ultimately periodic*

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- undecidable property
- classical proof: reduction from nilpotency
- pre-periods (t) are not coputable

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- pre-periods (t) are not coputable

▷ How large p can be?

▷ Can we decide if a CA is periodic ($\exists p, F^p = Id$)?

Equicontinuous CA

Large periods in 2D

Proposition

1. *Periodicity is undecidable in 2D*
2. *There is no recursive bound on* $\min(t, p)$ *for 2D CA*

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▷ Reduction from the infinite snake tiling problem

- a snake is a non-overlapping path in $\mathbb{Z}^2$
- formally, an injective function $s : \mathbb{N} \to \mathbb{Z}^2$ such that $s(n)$ and $s(n+1)$ are neighbours

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- a snake is a non-overlapping path in $\mathbb{Z}^2$
- formally, an injective function $s : \mathbb{N} \to \mathbb{Z}^2$ such that $s(n)$ and $s(n+1)$ are neighbours
- a wang tile set admits the snake s if the region $s(\mathbb{N})$ can be tiled correctly
- **infinite snake tiling problem:** given a tile set, does it admit an infinite snake?
- proven undecidable by Adleman *et al.*

Equicontinuous CA

Large periods in 2D (proof)

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- T a tile set
- add a component to have explicit path
  - $\{\uparrow, \downarrow, \rightarrow, \leftarrow\} \times \{\uparrow, \downarrow, \rightarrow, \leftarrow\}$
  - one arrow points to the successor
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- a configuration is made of pieces of valid snakes
- on each snake put a periodic dynamics
- bound on snake length $\iff$ globally periodic dynamics

□

Proposition

There is no expansive CA when $d \geq 2$.

- combinatorial proof
- **key idea:** there is no continuous function mapping a tube $Q^{[-k,k]^d \times \mathbb{N}}$ to the configuration space $Q^{\mathbb{Z}^d}$.

- **Problems:**
  - Cantor topology lacks of uniformity

    e.g.: σ is sensitive and Id is equicontinuous
  - Cantor balls only deal with a 0-dimensional zone of the cell lattice

    not enough expressiveness when $d \geq 2$

Directional dynamics

Intuitive ideas

- **Problems:**
  - Cantor topology lacks of uniformity

    e.g.: σ is sensitive and Id is equicontinuous
  - Cantor balls only deal with a 0-dimensional zone of the cell lattice

    not enough expressiveness when $d \geq 2$
- **A solution:**
  - consider action of both σ and F
  - consider other topologies which can deal with zones of the lattice of any dimension

Directional dynamics

Formalism

- restriction of the $\mathbb{Z}^d \times \mathbb{N}$-action $\big((\sigma_i), F\big)$:
  - $\mathbb{M}$ a semi-group $\subseteq \mathbb{Z}^d \times \mathbb{N}$
  - $\forall t$, $\mathbb{M}_t = \mathbb{M} \cap \mathbb{Z}^d \times \{t\} \neq \emptyset$
  - $\forall m \in \mathbb{M}$, $\mathcal{F}^m = \sigma_{\overrightarrow{v}} \circ F^t$ where $m = (\overrightarrow{v}, t)$

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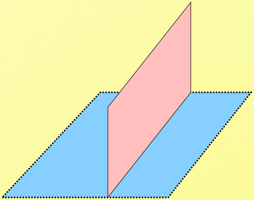
- typical example: a discretized vectorial subspace of $\mathbb{R}^d \times \mathbb{R}_+$



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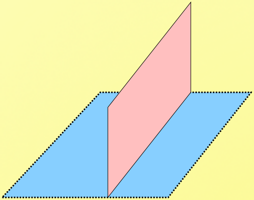

- transformation from topological to directional dynamics:

$$d(x,y) \leq \delta \quad \text{becomes} \quad \forall m \in \mathbb{M}_0, d(\mathcal{F}^m(x), \mathcal{F}^m(y)) \leq \delta$$
$$\forall/\exists t, d\big(F^t(x), F^t(y)\big) \quad \text{becomes} \quad \forall/\exists m \in \mathbb{M}, d\big(\mathcal{F}^m(x), \mathcal{F}^m(y)\big)$$

Directional dynamics

Classical properties revisited

- $\mathbb{M}$**-equicontinuity point** (at x):

$$\forall \epsilon, \exists \delta, \forall y : \forall m \in \mathbb{M}_0, d\big(\mathcal{F}^m(x), \mathcal{F}^m(y)\big) \leq \delta \Rightarrow \forall m \in \mathbb{M}, d\big(\mathcal{F}^m(x), \mathcal{F}^m(y)\big) \leq \epsilon$$

- $\mathbb{M}$**-sensitivity to initial conditions:**

$$\exists \epsilon, \forall x, \forall \delta > 0, \exists y : \forall m \in \mathbb{M}_0, d\big(\mathcal{F}^m(x), \mathcal{F}^m(y)\big) \leq \delta \text{ and } \exists m \in \mathbb{M}, d\big(\mathcal{F}^m(x), \mathcal{F}^m(y)\big) \geq \epsilon$$

- $\mathbb{M}$**-expansivity:**

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Directional dynamics
Properties

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Directional dynamics
Properties

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  - F not $\mathbb{M}$-sensitive $\iff$ F has $\mathbb{M}$-equicontinuity points
  - many results about slopes are still true:
    - equicontinuity implies rational slope
    - set of slopes with equicontinuity points is convex
  - there are $\mathbb{M}$-expansive CA:
    - permutative CA if neighbourhood compatible with $\mathbb{M}_0$
    - the set of expansivity slopes is open

Open questions

- Periods in 1D:
  - deciding if a CA is periodic
  - recursive bound on periods of equicontinuous CA
- Expansivity in 2D:
  - examples not based on permutative CA
  - decision problem easier to prove hard (undecidable)
- Domains of actions:
  - irrational slopes for equicontinuity points
  - general definition for domains
  - other examples needed